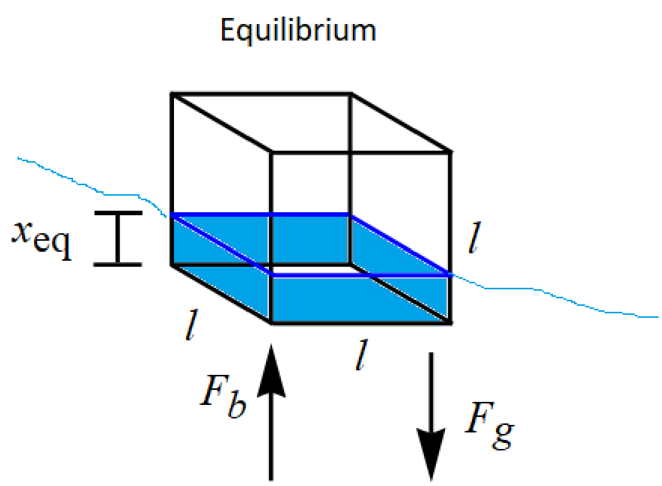
Problem 27

A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion and determine the period of the motion.
Hint: Use Archimedes' principle: an object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

Solution

Start by drawing a free-body diagram of the block. The two relevant forces are due to gravity and buoyancy.



And they balance each other in equilibrium.

$$F_b = F_g$$

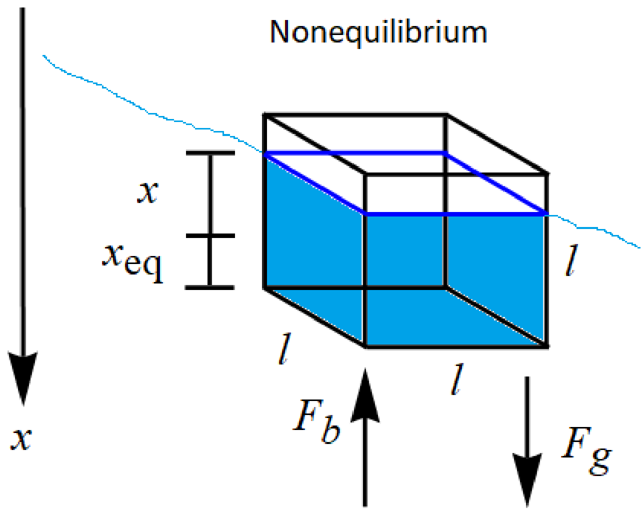
According to Archimedes's principle, the buoyant force is equal to the weight of the liquid displaced, so $F_b = (l^2 x_{eq})\rho_0 g$. The quantity in parentheses is the volume of liquid that the block takes up. Multiplying this result by ρ_0 gives the mass of this liquid, and further multiplying it by g gives its weight. On the other hand, F_g is the weight of the block: $F_g = mg = (l^3)\rho g$.

$$(l^2 x_{eq})\rho_0 g = (l^3)\rho g \tag{1}$$

Assume now that the block is submerged a distance x and then allowed to oscillate.

[10] Archimedes (287–212 BC) was the foremost of the ancient Greek mathematicians. He lived in Syracuse on the island of Sicily. His most notable discoveries were in geometry, but he also made important contributions to hydrostatics and other branches of mechanics. His method of exhaustion is a precursor of the integral calculus developed by Newton and Leibniz almost two millennia later. He died at the hands of a Roman soldier during the Second Punic War.

The free-body diagram in this case is shown below.



Apply Newton's second law in the x-direction to obtain the equation of motion.

$$\sum F_x = ma_x$$

$$-F_b + F_g = ma_x$$

Use the fact that acceleration is the second derivative of position and the fact that the buoyant force now is $F_b = \rho_0 l^2 (x_{\text{eq}} + x) g$.

$$-\rho_0 l^2 (x_{\text{eq}} + x) g + \rho l^3 g = mx''$$

Expand the left side.

$$-\cancel{\rho_0 l^2 x_{\text{eq}} g} - \rho_0 l^2 x g + \cancel{\rho l^3 g} = mx''$$

From equation (1), the first and third terms cancel.

$$-\rho_0 l^2 x g = mx''$$

Bring $\rho_0 l^2 x g$ to the right side.

$$mx'' + \rho_0 l^2 x g = 0$$

Divide both sides by m.

$$x'' + \frac{\rho_0 l^2 g}{m} x = 0$$

The mass of the block is its density multiplied by its volume: $m = \rho l^3$.

$$x'' + \frac{\rho_0 g}{\rho l} x = 0$$

The general solution can be written in terms of sine and cosine.

$$x(t) = C_1 \cos \sqrt{\frac{\rho_0 g}{\rho l}} t + C_2 \sin \sqrt{\frac{\rho_0 g}{\rho l}} t$$

C_1 and C_2 are arbitrary, so let $C_1 = A\cos\delta$ and $C_2 = A\sin\delta$ in order to combine the two sinusoidal terms into one.

$$\begin{aligned} x(t) &= A\cos\delta\cos\sqrt{\frac{\rho_0 g}{\rho l}}t + A\sin\delta\sin\sqrt{\frac{\rho_0 g}{\rho l}}t \\ &= A\cos\left(\sqrt{\frac{\rho_0 g}{\rho l}}t - \delta\right) \end{aligned}$$

We conclude that the angular frequency of oscillation is

$$\omega = \sqrt{\frac{\rho_0 g}{\rho l}}$$

and that the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\rho l}{\rho_0 g}}.$$