Problem 30

In the absence of damping, the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \qquad u(0) = a, \quad u'(0) = b.$$

(a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.

(b) Solve the given initial value problem.

(c) Using the solution in part (b), determine the total energy in the system at any time t. Your result should confirm the principle of conservation of energy for this system.

Solution

Part (a)

The potential energy for a spring is

$$U = \frac{1}{2}kx^2.$$

Initially the position of the mass is $u(0) = a$, so the potential energy at $t = 0$ is

$$U = \frac{1}{2}ka^2.$$

On the other hand, the kinetic energy is

$$T = \frac{1}{2}mv^2.$$

Initially the velocity of the mass is $u'(0) = b$, so the kinetic energy at $t = 0$ is

$$T = \frac{1}{2}mb^2.$$

At $t = 0$ then, the total mechanical energy is

$$E(0) = U + T = \frac{1}{2}ka^2 + \frac{1}{2}mb^2 = \frac{1}{2}(ka^2 + mb^2).$$

Part (b)

Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \to \quad u = re^{rt} \quad \to \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$m(r^2 e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m}$$

$$r = \left\{-i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}}\right\} = \{-i\omega, i\omega\}$$

Two solutions to the ODE are $u = e^{-i\omega t}$ and $u = e^{i\omega t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} u(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ &= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1\cos\omega t - iC_1\sin\omega t + C_2\cos\omega t + iC_2\sin\omega t \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t \end{aligned}$$

Differentiate it with respect to t.

$$u'(t) = -C_3\omega\sin\omega t + C_4\omega\cos\omega t$$

Apply the initial conditions here to determine C_3 and C_4.

$$\begin{aligned} u(0) &= C_3 = a \\ u'(0) &= C_4\omega = b \end{aligned}$$

Solving this system of equations yields $C_3 = a$ and $C_4 = b/\omega$. Therefore, the solution to the initial value problem is

$$\begin{aligned} u(t) &= a\cos\omega t + \frac{b}{\omega}\sin\omega t \\ &= a\cos\sqrt{\frac{k}{m}}t + b\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t. \end{aligned}$$

Part (c)

The total mechanical energy at any time is

$$\begin{aligned}
E(t) &= \frac{1}{2}ku^2 + \frac{1}{2}mu'^2 \\
&= \frac{1}{2}k\left(a\cos\sqrt{\frac{k}{m}}t + b\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t\right)^2 + \frac{1}{2}m\left(-a\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + b\cos\sqrt{\frac{k}{m}}t\right)^2 \\
&= \frac{1}{2}k\left(a^2\cos^2\sqrt{\frac{k}{m}}t + b^2\frac{m}{k}\sin^2\sqrt{\frac{k}{m}}t + 2ab\sqrt{\frac{m}{k}}\cos\sqrt{\frac{k}{m}}t\sin\sqrt{\frac{k}{m}}t\right) \\
&\qquad + \frac{1}{2}m\left(a^2\frac{k}{m}\sin^2\sqrt{\frac{k}{m}}t + b^2\cos^2\sqrt{\frac{k}{m}}t - 2ab\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t\cos\sqrt{\frac{k}{m}}t\right) \\
&= \frac{1}{2}\left(ka^2\cos^2\sqrt{\frac{k}{m}}t + mb^2\sin^2\sqrt{\frac{k}{m}}t + \cancel{2ab\sqrt{km}\cos\sqrt{\frac{k}{m}}t\sin\sqrt{\frac{k}{m}}t}\right. \\
&\qquad \left. + ka^2\sin^2\sqrt{\frac{k}{m}}t + mb^2\cos^2\sqrt{\frac{k}{m}}t - \cancel{2ab\sqrt{km}\sin\sqrt{\frac{k}{m}}t\cos\sqrt{\frac{k}{m}}t}\right) \\
&= \frac{1}{2}\left[ka^2\left(\cos^2\sqrt{\frac{k}{m}}t + \sin^2\sqrt{\frac{k}{m}}t\right) + mb^2\left(\sin^2\sqrt{\frac{k}{m}}t + \cos^2\sqrt{\frac{k}{m}}t\right)\right] \\
&= \frac{1}{2}(ka^2 + mb^2) \\
&= E(0).
\end{aligned}$$

The reason the total mechanical energy remains constant is that there is no damping.