Problem 11

A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position $u$ at any time $t$. Find the quasi frequency $\mu$ and the ratio of $\mu$ to the natural frequency of the corresponding undamped motion.

Solution

Use Hooke’s law to obtain the relationship between the spring force and the displacement.

$$F_s = k\Delta x$$

$$3 \text{ N} = k \left(10 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}}\right)$$

$$k = 30 \frac{\text{ N}}{\text{ m}}$$

Assume that the viscous damping force is proportional to the velocity.

$$F_d \propto x'$$

Introduce a proportionality constant $c$ to turn this into an equation we can use.

$$F_d = cx'$$

Use the fact that the damping force is 3 N when the velocity of the mass is 5 m/s to determine $c$.

$$3 \text{ N} = c \left(5 \frac{\text{ m}}{\text{ s}}\right)$$

$$c = 0.6 \frac{\text{ N} \cdot \text{ s}}{\text{ m}}$$

The aim now is to obtain an equation of motion. Start by drawing a free-body diagram of the mass.

www.stemjock.com
Apply Newton's second law in the $x$-direction.

$$\sum F_x = ma_x$$

$$-cx' - kx + mg = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-cx' - kx + mg = mx''$$

$$mx'' + cx' + kx = mg$$

This is a linear inhomogeneous ODE, so its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx'' + cx' + kx = 0$$ \hspace{1cm} (1)

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \rightarrow x'_c = re^{rt} \rightarrow x''_c = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for $r$.

$$m(r^2 e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Since $c^2 - 4mk < 0$, write the quantity under the square root as $4mk - c^2$.

$$r = \left\{ \frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m} \right\}$$

$$r = \left\{ \frac{-c}{2m} - i\mu, \frac{-c}{2m} + i\mu \right\}$$

Two solutions to equation (1) are $x_c = e^{(-c/2m-i\mu)t}$ and $x_c = e^{(-c/2m+i\mu)t}$. By the principle of superposition, the general solution for $x_c$ is a linear combination of the two.

$$x_c(t) = C_1 e^{(-c/2m-i\mu)t} + C_2 e^{(-c/2m+i\mu)t}$$

$$= C_1 e^{-ct/2m-i\mu t} + C_2 e^{-ct/2m+i\mu t}$$

$$= C_1 e^{-ct/2m} e^{-i\mu t} + C_2 e^{-ct/2m} e^{i\mu t}$$

$$= C_1 e^{-ct/2m} [\cos(-\mu t) + i \sin(-\mu t)] + C_2 e^{-ct/2m} [\cos(\mu t) + i \sin(\mu t)]$$

$$= C_1 e^{-ct/2m} [\cos(\mu t) - i \sin(\mu t)] + C_2 e^{-ct/2m} [\cos(\mu t) + i \sin(\mu t)]$$

$$= C_1 e^{-ct/2m} \cos \mu t - iC_1 e^{-ct/2m} \sin \mu t + C_2 e^{-ct/2m} \cos \mu t + iC_2 e^{-ct/2m} \sin \mu t$$

$$= (C_1 + C_2) e^{-ct/2m} \cos \mu t - i(C_1 - C_2) e^{-ct/2m} \sin \mu t$$

$$= C_3 e^{-ct/2m} \cos \mu t + C_4 e^{-ct/2m} \sin \mu t$$
On the other hand, the particular solution satisfies

\[ mx''_p + cx'_p + kx_p = mg. \]

Because the inhomogeneous term is a constant, the particular solution is a constant as well: \[ x_p(t) = A. \] Substitute this into the equation to determine \( A \).

\[ m(A)'' + c(A)' + k(A) = mg \]

\[ kA = mg \]

\[ A = \frac{mg}{k} \]

So then \( x_p(t) = \frac{mg}{k} \), which means that the general solution for \( x \) is

\[ x(t) = C_3 e^{-ct/2m} \cos \mu t + C_4 e^{-ct/2m} \sin \mu t + \frac{mg}{k}. \]

Take a derivative of it with respect to \( t \).

\[ x'(t) = -C_3 \frac{c}{2m} e^{-ct/2m} \cos \mu t - C_3 \mu e^{-ct/2m} \sin \mu t - C_4 \frac{c}{2m} e^{-ct/2m} \sin \mu t + C_4 \mu e^{-ct/2m} \cos \mu t \]

Now we will use initial conditions to find \( C_3 \) and \( C_4 \). To determine the first initial condition, \( x(0) \), we need to know the equilibrium height of the mass. Use Hooke’s law once again.

\[ mg = kx_{eq} \]

\[ x_{eq} = \frac{mg}{k} = \frac{(2 \text{ kg})(9.81 \text{ m/s}^2)}{30 \text{ N/m}} = 0.654 \text{ m} = 65.4 \text{ cm} \]

Consequently, the initial conditions are

\[ x(0) = x_{eq} + 5 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}} = \frac{mg}{k} + 0.05 \text{ m} \]

\[ x'(0) = 10 \text{ cm/s} \times \frac{1 \text{ m}}{100 \text{ cm}} = 0.1 \text{ m/s} \]

Apply them to get

\[ x(0) = C_3 + \frac{mg}{k} = \frac{mg}{k} + 0.05 \]

\[ x'(0) = -C_3 \frac{c}{2m} + C_4 \mu = 0.1. \]

Solving this system of equations yields

\[ C_3 = 0.05 \quad \text{and} \quad C_4 = \frac{1}{\mu} \left( 0.1 + \frac{0.05c}{2m} \right). \]

Therefore,

\[ x(t) = 0.05 e^{-ct/2m} \cos \mu t + \frac{1}{\mu} \left( 0.1 + \frac{0.05c}{2m} \right) e^{-ct/2m} \sin \mu t + \frac{mg}{k} \]

\[ = 0.05 e^{-0.15t} \cos \frac{\sqrt{4mk - c^2}}{2m} t + \frac{2m}{\sqrt{4mk - c^2}} \left( 0.1 + \frac{0.05c}{2m} \right) e^{-0.15t} \sin \frac{\sqrt{4mk - c^2}}{2m} t + \frac{mg}{k}. \]

www.stemjock.com
As $x(t)$ is in meters, multiply the result by 100 to get it in centimeters.

The quasi angular frequency is

$$\mu = \frac{\sqrt{4mk - c^2}}{2m} \approx 3.87 \text{ rad/s},$$

and the ratio of $\mu$ to the natural frequency of the corresponding undamped motion is

$$\frac{\mu}{\omega} = \frac{\sqrt{4mk - c^2}}{2m} \sqrt{\frac{k}{m}} \approx 0.99925.$$