Problem 12

A series circuit has a capacitor of $10^{-5}$ F, a resistor of $3 \times 10^2$ Ω, and an inductor of 0.2 H. The initial charge on the capacitor is $10^{-6}$ C and there is no initial current. Find the charge $Q$ on the capacitor at any time $t$.

Solution

Start by drawing an RLC series circuit.

![RLC circuit diagram]

Assume that the circuit is closed at $t = 0$. Apply Faraday’s law to obtain the governing ODE for the current.

$$\sum V = -L \frac{di}{dt}$$

The only potential drops occur over the resistor and the capacitor.

$$iR + \frac{q}{C} = -L \frac{di}{dt}$$

Write $i = dq/dt = q'$.

$$Rq' + \frac{q}{C} = -Lq''$$

Bring $Lq''$ to the left side and divide both sides by $L$.

$$q'' + \frac{R}{L} q' + \frac{1}{LC} q = 0 \quad (1)$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $q = e^{rt}$.

$$q = e^{rt} \rightarrow q' = re^{rt} \rightarrow q'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for $r$.

$$r^2 e^{rt} + \frac{R}{L}(re^{rt}) + \frac{1}{LC}(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$r^2 + \frac{R}{L} r + \frac{1}{LC} = 0$$

$$r = \frac{-R \pm \sqrt{R^2 - 4 LC}}{2} = -1500 \pm \frac{\sqrt{250000}}{2} = -750 \pm 250$$

$$r = \{-1000, -500\}$$

Two solutions to equation (1) are then $q = e^{-1000t}$ and $q = e^{-500t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$q(t) = C_1 e^{-1000t} + C_2 e^{-500t}$$

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Take a derivative of it with respect to $t$.

$$q'(t) = -1000C_1 e^{-1000t} - 500C_2 e^{-500t}$$

Now apply the initial conditions, $q(0) = 10^{-6}$ and $q'(0) = 0$, to determine $C_1$ and $C_2$.

$$q(0) = C_1 + C_2 = 10^{-6}$$
$$q'(0) = -1000C_1 - 500C_2 = 0$$

Solving this system of equations yields $C_1 = -10^{-6}$ and $C_2 = 2 \times 10^{-6}$. Therefore,

$$q(t) = -10^{-6}e^{-1000t} + 2 \times 10^{-6}e^{-500t}.$$ 

Note that the charge $q(t)$ is in Coulombs (C).