Problem 14

Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is \(2\pi\sqrt{L/g}\), where \(L\) is the elongation of the spring due to the mass, and \(g\) is the acceleration due to gravity.

Solution

Start by drawing a free-body diagram for the mass hanging from a spring.

![Free-body diagram](image)

Apply Newton’s second law to obtain the equation of motion.

\[
\sum F_x = ma_x
\]

The only two forces are due to the spring and gravity.

\[-kx + mg = ma_x\]

Use the fact that acceleration is the second derivative of position.

\[-kx + mg = mx''\]

\[mx'' + kx = mg\]

This is a linear inhomogeneous ODE, so the general solution can be written as a sum of the complementary solution and the particular solution.

\[x(t) = x_c(t) + x_p(t)\]

The complementary solution satisfies the associated homogeneous equation.

\[mx_c'' + kx_c = 0\] (1)

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form \(x_c = e^{rt}\).

\[x_c = e^{rt} \quad \rightarrow \quad x'_c = re^{rt} \quad \rightarrow \quad x''_c = r^2 e^{rt}\]
Substitute these expressions to obtain an algebraic equation for \( r \).
\[
m(r^2e^{rt}) + k(e^{rt}) = 0
\]
Divide both sides by \( e^{rt} \).
\[
mr^2 + k = 0
\]
\[
r^2 = -\frac{k}{m}
\]
\[
r = \left\{-i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}}\right\} = \{-i\omega, i\omega\}
\]
Two solutions to equation (1) are then \( x_c = e^{-i\omega t} \) and \( x_c = e^{i\omega t} \). By the principle of superposition, the general solution for \( x_c(t) \) is a linear combination of these two.
\[
x_c(t) = C_1 e^{-i\omega t} + C_2 e^{i\omega t}
\]
\[
= C_1 [\cos(-\omega t) + i \sin(-\omega t)] + C_2 [\cos(\omega t) + i \sin(\omega t)]
\]
\[
= C_1 [\cos(\omega t) - i \sin(\omega t)] + C_2 [\cos(\omega t) + i \sin(\omega t)]
\]
\[
= C_1 \cos\omega t - iC_1 \sin\omega t + C_2 \cos\omega t + iC_2 \sin\omega t
\]
\[
= (C_1 + C_2) \cos\omega t + (-iC_1 + iC_2) \sin\omega t
\]
\[
= C_3 \cos\omega t + C_4 \sin\omega t
\]
Since \( C_3 \) and \( C_4 \) are arbitrary, we can introduce an amplitude \( A \) and a phase \( \delta \) in order to write the two sinusoidal terms as one.
\[
x_c(t) = A \cos\delta \cos\omega t + A \sin\delta \sin\omega t
\]
\[
= A \cos(\omega t - \delta)
\]
\[
= A \cos \left( \sqrt{\frac{k}{m}} t - \delta \right)
\]
The period of the motion is
\[
T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi \sqrt{\frac{m}{k}}.
\]
In order to write this in terms of the elongation \( L \) and \( g \), we have to consider the spring in equilibrium.
Here the force due to gravity balances the force due to the spring.

\[ mg = kL \]

Solve this for \( m/k \).

\[ \frac{m}{k} = \frac{L}{g} \]

Therefore,

\[ T = 2\pi \sqrt{\frac{L}{g}}. \]