Problem 24

The position of a certain spring-mass system satisfies the initial value problem

\[ \frac{3}{2} u'' + ku = 0, \quad u(0) = 2, \quad u'(0) = v. \]

If the period and amplitude of the resulting motion are observed to be \( \pi \) and 3, respectively, determine the values of \( k \) and \( v \).

Solution

Multiply both sides of the ODE by 2/3.

\[ u'' + \frac{2k}{3}u = 0 \]

The general solution is

\[ u(t) = C_1 \cos \sqrt{\frac{2k}{3}} t + C_2 \sin \sqrt{\frac{2k}{3}} t. \]

Take a derivative of it with respect to \( t \).

\[ u'(t) = -C_1 \sqrt{\frac{2k}{3}} \sin \sqrt{\frac{2k}{3}} t + C_2 \sqrt{\frac{2k}{3}} \cos \sqrt{\frac{2k}{3}} t \]

Apply the initial conditions to determine \( C_1 \) and \( C_2 \).

\[ u(0) = C_1 = 2 \]

\[ u'(0) = C_2 \sqrt{\frac{2k}{3}} = v \quad \rightarrow \quad C_2 = v \sqrt{\frac{3}{2k}} \]

The solution to the initial value problem is then

\[ u(t) = 2 \cos \sqrt{\frac{2k}{3}} t + v \sqrt{\frac{3}{2k}} \sin \sqrt{\frac{2k}{3}} t. \]

Introduce an amplitude \( R \) and phase \( \delta \) to combine the two sinusoidal terms into one.

\[ u(t) = R \cos \delta \cos \sqrt{\frac{2k}{3}} t + R \sin \delta \sin \sqrt{\frac{2k}{3}} t \]

\[ = R \cos \left( \sqrt{\frac{2k}{3}} t - \delta \right) \]

\( R \) and \( \delta \) satisfy the following system of equations.

\[ R \cos \delta = 2 \]

\[ R \sin \delta = v \sqrt{\frac{3}{2k}} \]

Square both sides of each equation

\[ R^2 \cos^2 \delta = 4 \]

\[ R^2 \sin^2 \delta = v^2 \frac{3}{2k} \]

www.stemjock.com
and then add the respective sides to determine $R$.

$$R^2 \cos^2 \delta + R^2 \sin^2 \delta = 4 + \frac{v^2}{3} \frac{3}{2k}$$

$$R^2 = 4 + \frac{v^2}{2k} \frac{3}{2k}$$

$$R = \sqrt{4 + \frac{3v^2}{2k}}$$

On the other hand, the period of the motion is

$$T = \frac{2\pi}{\sqrt{\frac{2k}{3}}} = 2\pi \sqrt{\frac{3}{2k}}.$$ 

Use the fact that the period and amplitude are $\pi$ and 3, respectively, to determine the values of $k$ and $v$.

$$T = 2\pi \sqrt{\frac{3}{2k}} = \pi$$

$$R = \sqrt{4 + \frac{3v^2}{2k}} = 3$$

Solving this system of equations yields $k = 6$ and $v = \pm 2\sqrt{5}$.  

www.stemjock.com