Problem 26

Consider the initial value problem

\[ mu'' + \gamma u' + ku = 0, \quad u(0) = u_0, \quad u'(0) = v_0. \]

Assume that \( \gamma^2 < 4km \).

(a) Solve the initial value problem.

(b) Write the solution in the form \( u(t) = R\exp(-\gamma t/2m)\cos(\mu t - \delta) \). Determine \( R \) in terms of \( m, \gamma, k, u_0, \) and \( v_0 \).

(c) Investigate the dependence of \( R \) on the damping coefficient \( \gamma \) for fixed values of the other parameters.

Solution

Since the coefficients in the ODE are constant and the ODE is homogeneous, the solutions are of the form \( u = e^{rt} \).

\[ u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2e^{rt} \]

Substitute these expressions to obtain an algebraic equation for \( r \).

\[ m(r^2e^{rt}) + \gamma(re^{rt}) + k(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ mr^2 + \gamma r + k = 0 \]

\[ r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma i\sqrt{4mk - \gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\mu \]

\[ r = \left\{ -\frac{\gamma}{2m} - i\mu, -\frac{\gamma}{2m} + i\mu \right\} \]

Two solutions to the ODE are \( u = e^{-\gamma t/2m} \) and \( u = e^{-\gamma t/2m+i\mu} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ u(t) = C_1e^{-\gamma t/2m} + C_2e^{-\gamma t/2m+i\mu} \]

Differentiate it with respect to \( t \).

\[ u'(t) = -\frac{\gamma}{2m}e^{-\gamma t/2m}(C_3 \cos \mu t + C_4 \sin \mu t) + e^{-\gamma t/2m}(-C_3 \mu \sin \mu t + C_4 \mu \cos \mu t) \]

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Apply the initial conditions now to determine $C_3$ and $C_4$.

\[
\begin{align*}
  u(0) &= C_3 = u_0 \\
  u'(0) &= -\frac{\gamma}{2m} C_3 + C_4 \mu = v_0
\end{align*}
\]

Solving this system of equations yields

\[
C_3 = u_0 \quad \text{and} \quad C_4 = \frac{1}{\mu} \left( v_0 + \frac{\gamma}{2m} u_0 \right).
\]

The solution to the initial value problem is then

\[
\begin{align*}
  u(t) &= e^{-\gamma t/2m} \left[ u_0 \cos \mu t + \frac{1}{\mu} \left( v_0 + \frac{\gamma}{2m} u_0 \right) \sin \mu t \right] \\
  &= e^{-\gamma t/2m} \left[ u_0 \cos \frac{\sqrt{4mk - \gamma^2}}{2m} t + \frac{2m}{\sqrt{4mk - \gamma^2}} \left( v_0 + \frac{\gamma}{2m} u_0 \right) \sin \frac{\sqrt{4mk - \gamma^2}}{2m} t \right] \\
  &= e^{-\gamma t/2m} \left[ u_0 \cos \frac{\sqrt{4mk - \gamma^2}}{2m} t + \frac{1}{\sqrt{4mk - \gamma^2}} (2mv_0 + \gamma u_0) \sin \frac{\sqrt{4mk - \gamma^2}}{2m} t \right].
\end{align*}
\]

Now introduce an amplitude $R$ and a phase $\delta$ that satisfy

\[
\begin{align*}
  R \cos \delta &= u_0 \\
  R \sin \delta &= \frac{1}{\sqrt{4mk - \gamma^2}} (2mv_0 + \gamma u_0)
\end{align*}
\]

so that the solution becomes

\[
\begin{align*}
  u(t) &= e^{-\gamma t/2m} \left[ R \cos \delta \cos \frac{\sqrt{4mk - \gamma^2}}{2m} t + R \sin \delta \sin \frac{\sqrt{4mk - \gamma^2}}{2m} t \right] \\
  &= Re^{-\gamma t/2m} \cos \left( \frac{\sqrt{4mk - \gamma^2}}{2m} t - \delta \right).
\end{align*}
\]

Solving the system of equations for $R$ and $\delta$ yields

\[
R = \sqrt{u_0^2 + \frac{(2mv_0 + \gamma u_0)^2}{4km - \gamma^2}} \quad \text{and} \quad \tan \delta = \frac{1}{u_0 \sqrt{4mk - \gamma^2}} (2mv_0 + \gamma u_0).
\]
Below is a plot of $R$ versus $\gamma$ for $u_0 = 1$, $m = 1$, $v_0 = 1$, and $k = 1$. 

\[ R(\gamma) = \sqrt{1 + \frac{(2 + \gamma)^2}{4 - \gamma^2}} \]