Problem 28

The position of a certain undamped spring-mass system satisfies the initial value problem

\[ u'' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2. \]

(a) Find the solution of this initial value problem.

(b) Plot \( u \) versus \( t \) and \( u' \) versus \( t \) on the same axes.

(c) Plot \( u' \) versus \( u \); that is, plot \( u(t) \) and \( u'(t) \) parametrically with \( t \) as the parameter. This plot is known as a phase plot, and the \( uu' \)-plane is called the phase plane. Observe that a closed curve in the phase plane corresponds to a periodic solution \( u(t) \). What is the direction of motion on the phase plot as \( t \) increases?

Solution

Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form \( u = e^{rt} \).

\[ u = e^{rt} \quad \rightarrow \quad u = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2 e^{rt} + 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 2 = 0 \]

\[ r = \{ -i\sqrt{2}, i\sqrt{2} \} \]

Two solutions to the ODE are \( u = e^{-i\sqrt{2}t} \) and \( u = e^{i\sqrt{2}t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[
\begin{align*}
  u(t) &= C_1 e^{-i\sqrt{2}t} + C_2 e^{i\sqrt{2}t} \\
  &= C_1 [\cos(-\sqrt{2}t) + i \sin(-\sqrt{2}t)] + C_2 [\cos(\sqrt{2}t) + i \sin(\sqrt{2}t)] \\
  &= C_1 [\cos(\sqrt{2}t) - i \sin(\sqrt{2}t)] + C_2 [\cos(\sqrt{2}t) + i \sin(\sqrt{2}t)] \\
  &= C_1 \cos \sqrt{2}t - iC_1 \sin \sqrt{2}t + C_2 \cos \sqrt{2}t + iC_2 \sin \sqrt{2}t \\
  &= (C_1 + C_2) \cos \sqrt{2}t + (-iC_1 + iC_2) \sin \sqrt{2}t \\
  &= C_3 \cos \sqrt{2}t + C_4 \sin \sqrt{2}t
\end{align*}
\]

Differentiate it with respect to \( t \).

\[
\begin{align*}
  u'(t) &= -C_3 \sqrt{2} \sin \sqrt{2}t + C_4 \sqrt{2} \cos \sqrt{2}t
\end{align*}
\]

Apply the initial conditions here to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
  u(0) &= C_3 = 0 \\
  u'(0) &= C_4 \sqrt{2} = 2
\end{align*}
\]

Solving this system of equations yields \( C_3 = 0 \) and \( C_4 = \sqrt{2} \). Therefore, the solution to the initial value problem is

\[ u(t) = \sqrt{2} \sin \sqrt{2}t. \]
\begin{align*}
x &= u(t) = \sqrt{2} \sin(\sqrt{2} t) \\
x' &= u'(t) = 2 \cos(\sqrt{2} t)
\end{align*}