Problem 30

In the absence of damping, the motion of a spring-mass system satisfies the initial value problem

\[ mu'' + ku = 0, \quad u(0) = a, \quad u'(0) = b. \]

(a) Show that the kinetic energy initially imparted to the mass is \( mb^2 / 2 \) and that the potential energy initially stored in the spring is \( ka^2 / 2 \), so that initially the total energy in the system is \( (ka^2 + mb^2) / 2 \).

(b) Solve the given initial value problem.

(c) Using the solution in part (b), determine the total energy in the system at any time \( t \). Your result should confirm the principle of conservation of energy for this system.

Solution

Part (a)

The potential energy for a spring is

\[ U = \frac{1}{2} kx^2. \]

Initially the position of the mass is \( u(0) = a \), so the potential energy at \( t = 0 \) is

\[ U = \frac{1}{2} ka^2. \]

On the other hand, the kinetic energy is

\[ T = \frac{1}{2} mv^2. \]

Initially the velocity of the mass is \( u'(0) = b \), so the kinetic energy at \( t = 0 \) is

\[ T = \frac{1}{2} mb^2. \]

At \( t = 0 \) then, the total mechanical energy is

\[ E(0) = U + T = \frac{1}{2} ka^2 + \frac{1}{2} mb^2 = \frac{1}{2} (ka^2 + mb^2). \]
Part (b)

Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form 
\[ u = e^{rt}. \]

\[ u = e^{rt} \quad \rightarrow \quad u = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ m(r^2 e^{rt}) + k(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ mr^2 + k = 0 \]

\[ r^2 = -\frac{k}{m} \]

\[ r = \left\{ -i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}} \right\} = \{-i\omega, i\omega\} \]

Two solutions to the ODE are \( u = e^{-i\omega t} \) and \( u = e^{i\omega t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ u(t) = C_1 e^{-i\omega t} + C_2 e^{i\omega t} \]

\[ = C_1 [\cos(-\omega t) + i \sin(-\omega t)] + C_2 [\cos(\omega t) + i \sin(\omega t)] \]

\[ = C_1 [\cos(\omega t) - i \sin(\omega t)] + C_2 [\cos(\omega t) + i \sin(\omega t)] \]

\[ = C_1 \cos \omega t - iC_1 \sin \omega t + C_2 \cos \omega t + iC_2 \sin \omega t \]

\[ = (C_1 + C_2) \cos \omega t + (-iC_1 + iC_2) \sin \omega t \]

\[ = C_3 \cos \omega t + C_4 \sin \omega t \]

Differentiate it with respect to \( t \).

\[ u'(t) = -C_3 \omega \sin \omega t + C_4 \omega \cos \omega t \]

Apply the initial conditions here to determine \( C_3 \) and \( C_4 \).

\[ u(0) = C_3 = a \]

\[ u'(0) = C_4 \omega = b \]

Solving this system of equations yields \( C_3 = a \) and \( C_4 = b/\omega \). Therefore, the solution to the initial value problem is

\[ u(t) = a \cos \omega t + \frac{b}{\omega} \sin \omega t \]

\[ = a \cos \sqrt{\frac{k}{m}} t + b \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}} t. \]
Part (c)

The total mechanical energy at any time is

\[ E(t) = \frac{1}{2}ku^2 + \frac{1}{2}mu'^2 \]

\[ = \frac{1}{2}k \left( a \cos \sqrt{\frac{k}{m}}t + b \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}}t \right)^2 + \frac{1}{2}m \left( -a \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t + b \cos \sqrt{\frac{k}{m}}t \right)^2 \]

\[ = \frac{1}{2}k \left( a^2 \cos^2 \sqrt{\frac{k}{m}}t + b^2 \frac{m}{k} \sin^2 \sqrt{\frac{k}{m}}t + 2ab \sqrt{\frac{m}{k}} \cos \sqrt{\frac{k}{m}}t \sin \sqrt{\frac{k}{m}}t \right) \]

\[ + \frac{1}{2}m \left( a^2 \frac{k}{m} \sin^2 \sqrt{\frac{k}{m}}t + b^2 \cos^2 \sqrt{\frac{k}{m}}t - 2ab \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t \cos \sqrt{\frac{k}{m}}t \right) \]

\[ = \frac{1}{2} \left( ka^2 \cos^2 \sqrt{\frac{k}{m}}t + mb^2 \sin^2 \sqrt{\frac{k}{m}}t + 2ab \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}}t \sin \sqrt{\frac{k}{m}}t \right) \]

\[ + ka^2 \sin^2 \sqrt{\frac{k}{m}}t + mb^2 \cos^2 \sqrt{\frac{k}{m}}t - 2ab \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t \cos \sqrt{\frac{k}{m}}t \right) \]

\[ = \frac{1}{2} \left[ ka^2 \left( \cos^2 \sqrt{\frac{k}{m}}t + \sin^2 \sqrt{\frac{k}{m}}t \right) + mb^2 \left( \sin^2 \sqrt{\frac{k}{m}}t + \cos^2 \sqrt{\frac{k}{m}}t \right) \right] \]

\[ = \frac{1}{2}(ka^2 + mb^2) \]

\[ = E(0). \]

The reason the total mechanical energy remains constant is that there is no damping.