Problem 32

In the spring-mass system of Problem 31, suppose that the spring force is not given by Hooke’s law but instead satisfies the relation

\[ F_s = -(ku + \epsilon u^3), \]

where \( k > 0 \) and \( \epsilon \) is small but may be of either sign. The spring is called a hardening spring if \( \epsilon > 0 \) and a softening spring if \( \epsilon < 0 \). Why are these terms appropriate?

(a) Show that the displacement \( u(t) \) of the mass from its equilibrium position satisfies the differential equation

\[ mu'' + \gamma u' + ku + \epsilon u^3 = 0. \]

Suppose that the initial conditions are

\[ u(0) = 0, \quad u'(0) = 1. \]

In the remainder of this problem, assume that \( m = 1, k = 1, \) and \( \gamma = 0. \)

(b) Find \( u(t) \) when \( \epsilon = 0 \) and also determine the amplitude and period of the motion.

(c) Let \( \epsilon = 0.1 \). Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.

(d) Repeat part (c) for \( \epsilon = 0.2 \) and \( \epsilon = 0.3 \).

(e) Plot your estimated values of the amplitude \( A \) and the period \( T \) versus \( \epsilon \). Describe the way in which \( A \) and \( T \), respectively, depend on \( \epsilon \).

(f) Repeat parts (c), (d), and (e) for negative values of \( \epsilon \).