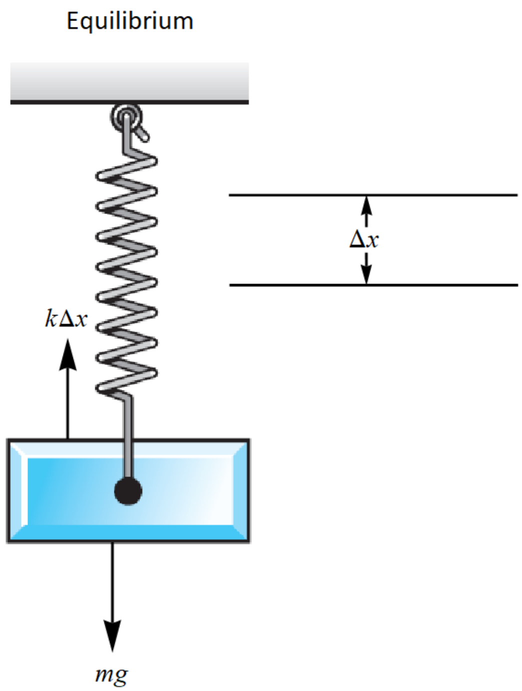
Problem 6

A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.



The gravitational and spring forces balance each other.

$$mg = k\Delta x$$

From this equation, k can be determined.

$$(5\text{ kg})\left(9.81\ \frac{\text{m}}{\text{s}^2}\right) = k\left(10\ \cancel{\text{cm}} \times \frac{1\text{ m}}{100\ \cancel{\text{cm}}}\right)$$

$$k = 490.5\ \frac{\text{N}}{\text{m}}$$

Consider now the mass in nonequilibrium. Assume that the damping force F_d is proportional to the speed of the mass.

$$F_d \propto x'$$

Introduce a proportionality constant c to change this to an equation.

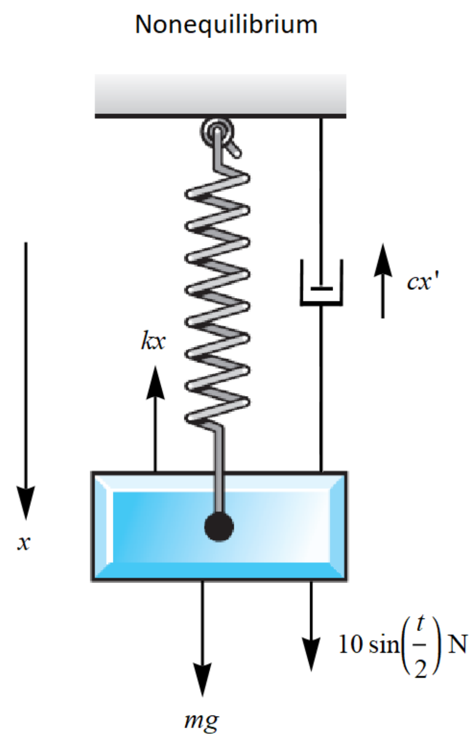
$$F_d = cx'$$

Given that the damping force is 2 N when the speed is 4 cm/s, c can be determined.

$$2\text{ N} = c\left(4\,\frac{\cancel{\text{cm}}}{\text{s}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}}\right)$$

$$c = 50\,\frac{\text{N}\cdot\text{s}}{\text{m}}$$

Draw a free-body diagram for the mass in nonequilibrium.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-cx' - kx + mg + 10\sin\frac{t}{2} = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-cx' - kx + mg + 10\sin\frac{t}{2} = mx''$$

$$mx'' + cx' + kx = mg + 10\sin\frac{t}{2}$$

Plug in the values for the constants.

$$5x'' + 50x' + 490.5x = 5g + 10\sin\frac{t}{2}$$

Divide both sides by 5.

$$x'' + 10x' + 98.1x = g + 2\sin\frac{t}{2}$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$x_c'' + 10x_c' + 98.1x_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + 10(re^{rt}) + 98.1(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 10r + 98.1 = 0$$

$$r = \frac{-10 \pm \sqrt{100 - 4(98.1)}}{2} = \frac{-10 \pm \sqrt{-292.4}}{2} = -5 \pm i\frac{\sqrt{292.4}}{2} = -5 \pm i\mu$$

Two solutions to equation (1) are then

$$x_c = e^{(-5-i\mu)t} \quad \text{and} \quad x_c = e^{(-5+i\mu)t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned}
x_c(t) &= C_1e^{(-5-i\mu)t} + C_2e^{(-5+i\mu)t} \\
&= C_1e^{-5t-i\mu t} + C_2e^{-5t+i\mu t} \\
&= C_1e^{-5t}e^{-i\mu t} + C_2e^{-5t}e^{i\mu t} \\
&= C_1e^{-5t}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-5t}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-5t}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-5t}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-5t}\cos\mu t - iC_1e^{-5t}\sin\mu t + C_2e^{-5t}\cos\mu t + iC_2e^{-5t}\sin\mu t \\
&= (C_1 + C_2)e^{-5t}\cos\mu t + (-iC_1 + iC_2)e^{-5t}\sin\mu t \\
&= C_3e^{-5t}\cos\mu t + C_4e^{-5t}\sin\mu t \\
&= e^{-5t}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

On the other hand, the particular solution satisfies

$$x_p'' + 10x_p' + 98.1x_p = g + 2\sin\frac{t}{2}.$$

For the first term on the right side, we'll include a constant A in the trial solution. Also, since there are even and odd derivatives on the left side, we'll include $B\cos(t/2) + C\sin(t/2)$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\cos(t/2) + C\sin(t/2)$ into the equation to determine A and B and C.

$$\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right)'' + 10\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right)' + 98.1\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right) = g + 2\sin\frac{t}{2}$$

$$\left(-\frac{B}{4}\cos\frac{t}{2} - \frac{C}{4}\sin\frac{t}{2}\right) + 10\left(-\frac{B}{2}\sin\frac{t}{2} + \frac{C}{2}\cos\frac{t}{2}\right) + 98.1\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right) = g + 2\sin\frac{t}{2}$$

$$98.1A + (97.85B + 5C)\cos\frac{t}{2} + (-5B + 97.85C)\sin\frac{t}{2} = g + 2\sin\frac{t}{2}$$

Matching the coefficients, we have

$$\begin{aligned} 98.1A &= g \\ 97.85B + 5C &= 0 \\ -5B + 97.85C &= 2. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{10} = 0.1 \quad \text{and} \quad B = -\frac{4000}{3\,839\,849} \quad \text{and} \quad C = \frac{78\,280}{3\,839\,849}.$$

The particular solution is then

$$x_p(t) = \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2},$$

which means the general solution is

$$\begin{aligned} x(t) &= e^{-5t}(C_3\cos\mu t + C_4\sin\mu t) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2} \\ &= e^{-5t}\left(C_3\cos\frac{\sqrt{292.4}}{2}t + C_4\sin\frac{\sqrt{292.4}}{2}t\right) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2} \\ &= e^{-5t}(C_3\cos\sqrt{73.1}t + C_4\sin\sqrt{73.1}t) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}. \end{aligned}$$

Take a derivative of it with respect to t.

$$\begin{aligned} x'(t) = -5e^{-5t}(C_3\cos\sqrt{73.1}t + C_4\sin\sqrt{73.1}t) + e^{-5t}(-\sqrt{73.1}C_3\sin\sqrt{73.1}t + C_4\sqrt{73.1}\cos\sqrt{73.1}t) \\ + \frac{2000}{3\,839\,849}\sin\frac{t}{2} + \frac{39\,140}{3\,839\,849}\cos\frac{t}{2} \end{aligned}$$

Now apply the initial conditions,

$$\begin{aligned} x(0) &= 10\,\cancel{\text{cm}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}} = \frac{1}{10}\text{ m} \\ x'(0) &= 3\,\frac{\cancel{\text{cm}}}{\text{s}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}} = \frac{3}{100}\,\frac{\text{m}}{\text{s}}, \end{aligned}$$

to determine C_3 and C_4.

$$\begin{aligned} x(0) &= C_3 + \frac{1}{10} - \frac{4000}{3\,839\,849} = \frac{1}{10} \\ x'(0) &= -5C_3 + C_4\sqrt{73.1} + \frac{39\,140}{3\,839\,849} = \frac{3}{100} \end{aligned}$$

Solving this system of equations yields

$$C_3 = \frac{4000}{3\,839\,849} \quad \text{and} \quad C_4 = \frac{9\,605\,547}{383\,984\,900\sqrt{73.1}},$$

so

$$x(t) = e^{-5t}\left(\frac{4000}{3\,839\,849}\cos\sqrt{73.1}t + \frac{9\,605\,547}{383\,984\,900\sqrt{73.1}}\sin\sqrt{73.1}t\right) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}.$$

As $x(t)$ is in meters, multiply the result by 100 to convert it to centimeters.