

Problem 18

Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3 \cos \omega t, \quad u(0) = 0, \quad u'(0) = 0.$$

- (a) Find the solution $u(t)$ for $\omega \neq 1$.
- (b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

Solution

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + u_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = r e^{rt} \quad \rightarrow \quad u_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r .

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are $u_c = e^{-it}$ and $u_c = e^{it}$. By the principle of superposition, the general solution for u_c is a linear combination of these two.

$$\begin{aligned} u_c(t) &= C_1 e^{-it} + C_2 e^{it} \\ &= C_1 [\cos(-t) + i \sin(-t)] + C_2 [\cos(t) + i \sin(t)] \\ &= C_1 [\cos(t) - i \sin(t)] + C_2 [\cos(t) + i \sin(t)] \\ &= C_1 \cos t - i C_1 \sin t + C_2 \cos t + i C_2 \sin t \\ &= (C_1 + C_2) \cos t + (-i C_1 + i C_2) \sin t \\ &= C_3 \cos t + C_4 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + u_p = 3 \cos \omega t.$$

Since there are only even derivatives on the left side, include only cosine in the trial solution to account for the cosine term on the right side: $u_p(t) = A \cos \omega t$. Plug this into the ODE to determine A .

$$(A \cos \omega t)'' + (A \cos \omega t) = 3 \cos \omega t$$

Evaluate the derivatives.

$$\begin{aligned}(-A\omega^2 \cos \omega t) + (A \cos \omega t) &= 3 \cos \omega t \\(-A\omega^2 + A) \cos \omega t &= 3 \cos \omega t\end{aligned}$$

Match the coefficients.

$$-A\omega^2 + A = 3$$

Solving this equation for A yields

$$A = \frac{3}{1 - \omega^2}.$$

Consequently, the particular solution is

$$u_p(t) = \frac{3}{1 - \omega^2} \cos \omega t,$$

and the general solution is

$$u(t) = C_3 \cos t + C_4 \sin t + \frac{3}{1 - \omega^2} \cos \omega t.$$

Differentiate it with respect to t .

$$u'(t) = -C_3 \sin t + C_4 \cos t - \omega \frac{3}{1 - \omega^2} \sin \omega t.$$

Now apply the initial conditions to determine C_3 and C_4 .

$$\begin{aligned}u(0) &= C_3 + \frac{3}{1 - \omega^2} = 0 \\u'(0) &= C_4 = 0\end{aligned}$$

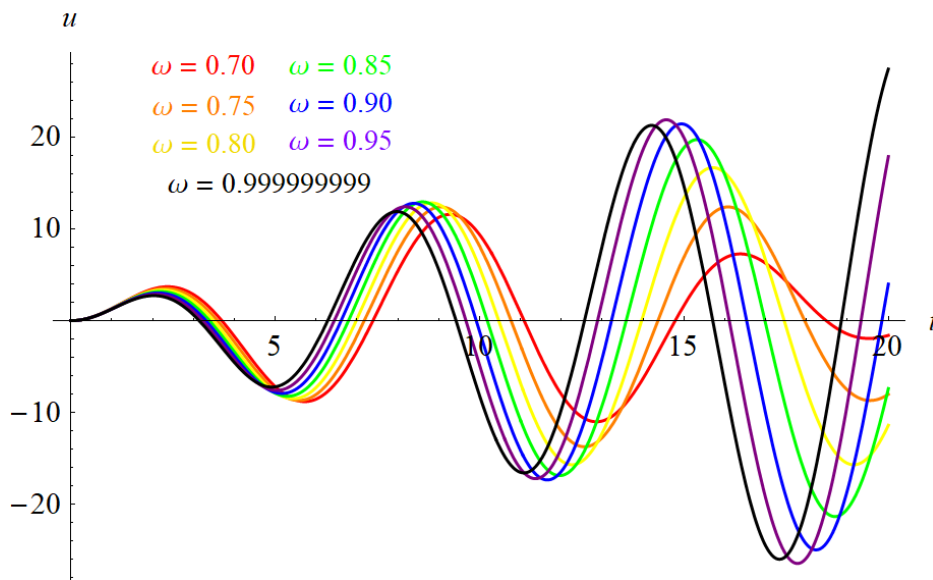
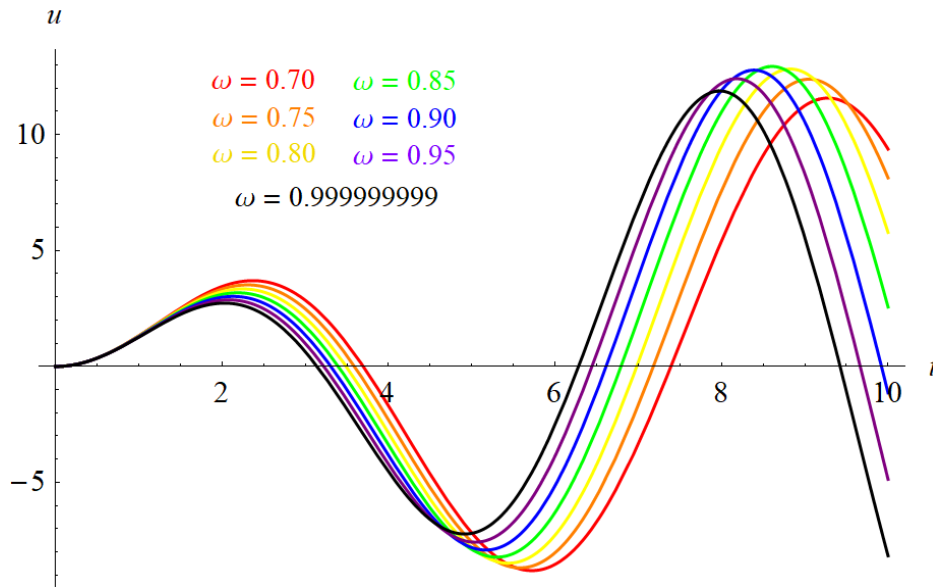
Solving this system of equations yields

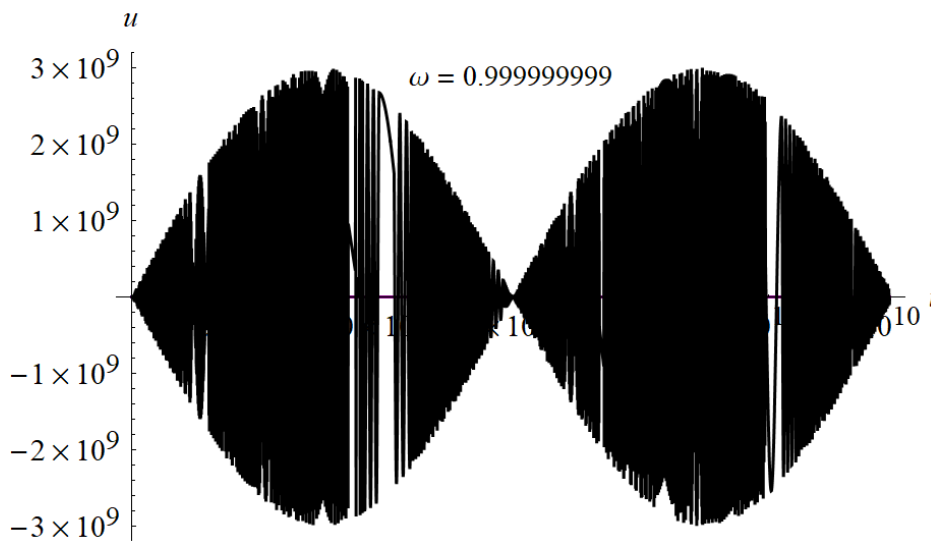
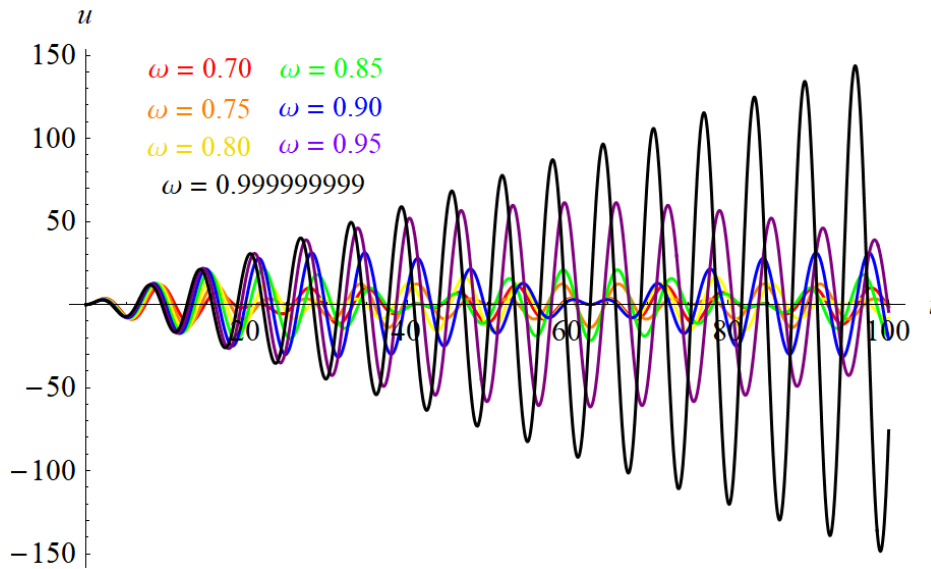
$$C_3 = -\frac{3}{1 - \omega^2} \quad \text{and} \quad C_4 = 0.$$

Therefore, assuming $\omega \neq 1$,

$$\begin{aligned}u(t) &= -\frac{3}{1 - \omega^2} \cos t + \frac{3}{1 - \omega^2} \cos \omega t \\&= \frac{3}{1 - \omega^2} (\cos \omega t - \cos t).\end{aligned}$$

Below are plots of $u(t)$ versus t for various values of ω and various scales of time.





Notice that the amplitude appears to grow linearly over some amount of time for each graph. The closer ω is to 1, the longer the amplitude will grow before finally oscillating back. If $\omega = 1$, then no such oscillation occurs.