Problem 19

Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos\omega t, \qquad u(0) = 1, \quad u'(0) = 1.$$

(a) Find the solution for $\omega \neq 1$.

(b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 18; that is, describe the effect of the nonzero initial conditions.

Solution

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + u_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are $u_c = e^{-it}$ and $u_c = e^{it}$. By the principle of superposition, the general solution for u_c is a linear combination of these two.

$$\begin{aligned} u_c(t) &= C_1e^{-it} + C_2e^{it} \\ &= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\ &= C_3\cos t + C_4\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + u_p = 3\cos\omega t.$$

Since there are only even derivatives on the left side, include only cosine in the trial solution to account for the cosine term on the right side: $u_p(t) = A\cos\omega t$. Plug this into the ODE to determine A.

$$(A\cos\omega t)'' + (A\cos\omega t) = 3\cos\omega t$$

Evaluate the derivatives.

$$(-A\omega^2 \cos \omega t) + (A \cos \omega t) = 3 \cos \omega t$$
$$(-A\omega^2 + A) \cos \omega t = 3 \cos \omega t$$

Match the coefficients.

$$-A\omega^2 + A = 3$$

Solving this equation for A yields

$$A = \frac{3}{1 - \omega^2}.$$

Consequently, the particular solution is

$$u_p(t) = \frac{3}{1 - \omega^2} \cos \omega t,$$

and the general solution is

$$u(t) = C_3 \cos t + C_4 \sin t + \frac{3}{1 - \omega^2} \cos \omega t.$$

Differentiate it with respect to t.

$$u'(t) = -C_3 \sin t + C_4 \cos t - \omega \frac{3}{1 - \omega^2} \sin \omega t.$$

Now apply the initial conditions to determine C_3 and C_4.

$$u(0) = C_3 + \frac{3}{1 - \omega^2} = 1$$
$$u'(0) = C_4 = 1$$

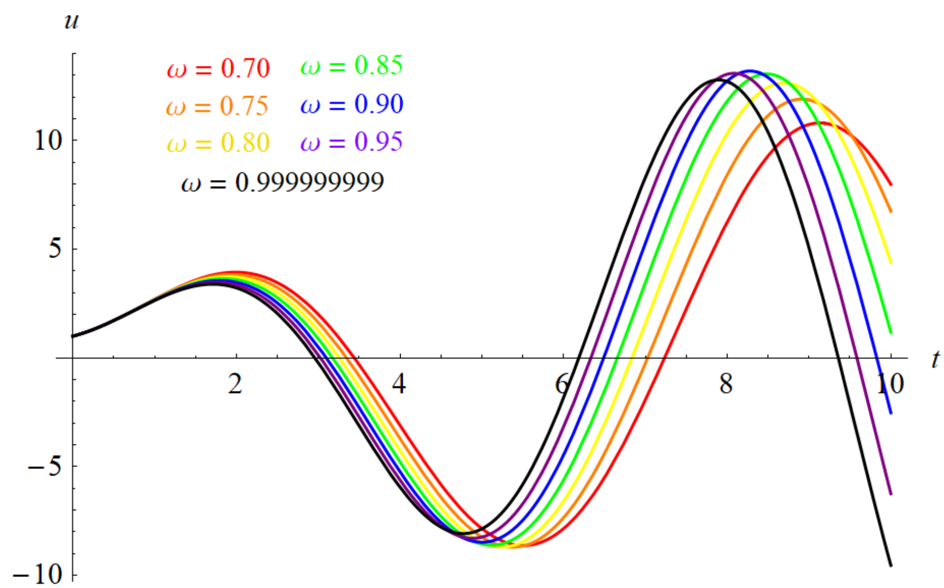
Solving this system of equations yields

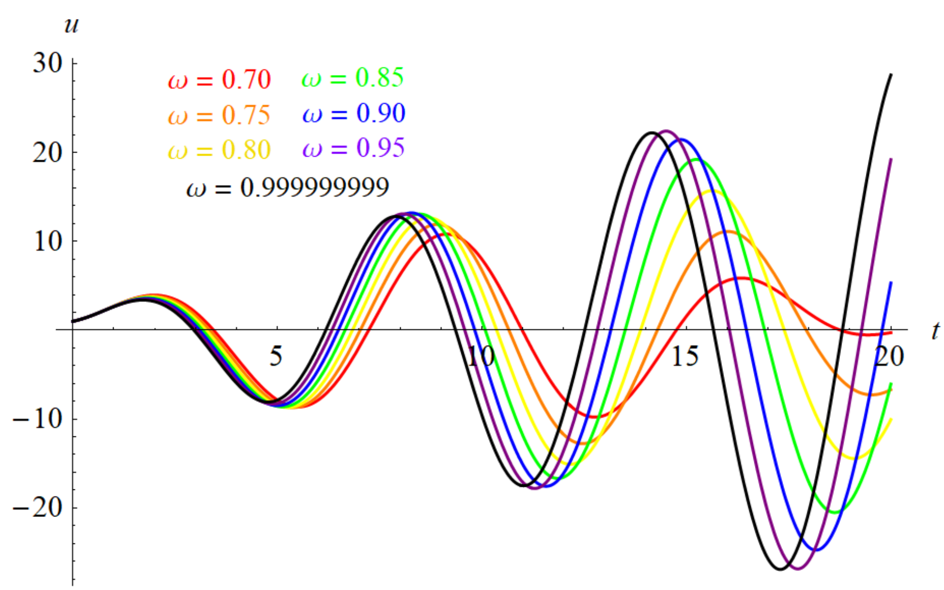
$$C_3 = \frac{\omega^2 + 2}{\omega^2 - 1} \quad \text{and} \quad C_4 = 1.$$

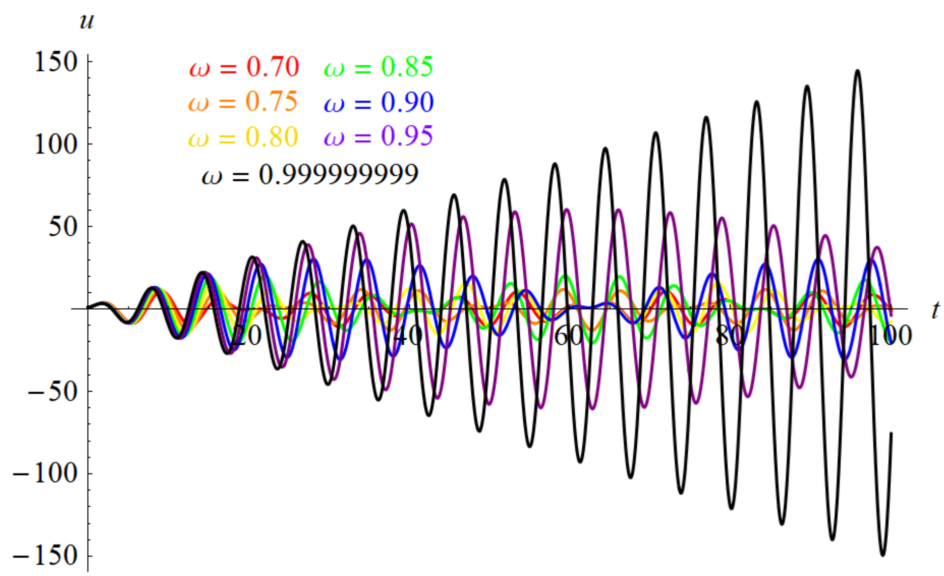
Therefore, assuming $\omega \neq 1$,

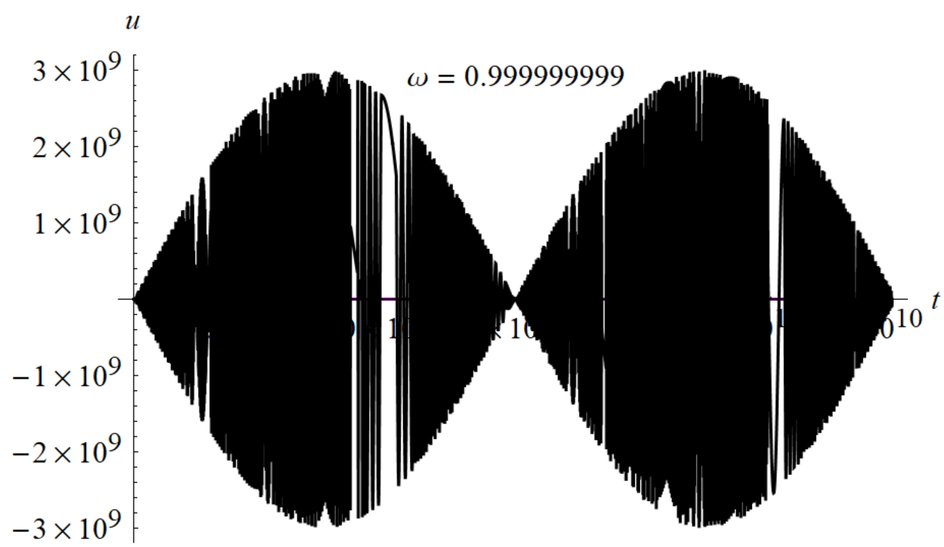
$$u(t) = \frac{\omega^2 + 2}{\omega^2 - 1} \cos t + \sin t + \frac{3}{1 - \omega^2} \cos \omega t.$$

Below are plots of $u(t)$ versus t for various values of ω and various scales of time.


u
t
ω = 0.70
ω = 0.85
ω = 0.75
ω = 0.90
ω = 0.80
ω = 0.95
ω = 0.999999999
10
5
−5
−10
2
4
6
8
10


u
t
ω = 0.70
ω = 0.85
ω = 0.75
ω = 0.90
ω = 0.80
ω = 0.95
ω = 0.999999999
30
20
10
−10
−20
5
10
15
20





Notice that the amplitude appears to grow linearly over some amount of time for each graph. The closer ω is to 1, the longer the amplitude will grow before finally oscillating back. If $\omega = 1$, then no such oscillation occurs. The nonzero initial conditions don't seem to change anything.