Problem 10

A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8 \sin 8t$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.

The gravitational and spring forces balance each other.

$$W = k \Delta x$$

From this equation, $k$ can be determined.

$$8 \text{ lb} = k(6 \text{ in})$$

$$k = \frac{4 \text{ lb}}{3 \text{ in}} \times \frac{12 \text{ in}}{1 \text{ ft}} = 16 \text{ lb ft}$$
Consider now the mass in nonequilibrium.

Apply Newton’s second law in the $x$-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-kx + W + 8 \sin 8t = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$m \dddot{x} + kx = W + 8 \sin 8t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$m \dddot{x}_c + kx_c = 0 \quad (1)$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x'_c = re^{rt} \quad \rightarrow \quad x''_c = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for $r$.

$$m(r^2 e^{rt}) + k(e^{rt}) = 0$$

www.stemjock.com
Divide both sides by $e^{rt}$.

\[
mr^2 + k = 0
\]

\[
r^2 = -\frac{k}{m}
\]

\[
r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega
\]

Two solutions to equation (1) are then

\[
x_c = e^{-i\omega t} \quad \text{and} \quad x_c = e^{i\omega t}.
\]

By the principle of superposition, the general solution for $x_c$ is a linear combination of these two.

\[
x_c(t) = C_1e^{-i\omega t} + C_2e^{i\omega t}
\]

\[
= C_1[\cos(-\omega t) + i \sin(-\omega t)] + C_2[\cos(\omega t) + i \sin(\omega t)]
\]

\[
= C_1[\cos(\omega t) - i \sin(\omega t)] + C_2[\cos(\omega t) + i \sin(\omega t)]
\]

\[
= C_1 \cos \omega t - iC_1 \sin \omega t + C_2 \cos \omega t + iC_2 \sin \omega t
\]

\[
= (C_1 + C_2) \cos \omega t + (-iC_1 + iC_2) \sin \omega t
\]

\[
= C_3 \cos \omega t + C_4 \sin \omega t
\]

On the other hand, the particular solution satisfies

\[
mx''_p + kx_p = W + 8 \sin 8t.
\]

For the first term on the right side, we’ll include a constant $A$ in the trial solution. Also, since there are only even derivatives on the left side, we’ll include $B \sin 8t$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B \sin 8t$ into the equation to determine $A$ and $B$.

\[
m(A + B \sin 8t)'' + k(A + B \sin 8t) = W + 8 \sin 8t
\]

\[
m(-64B \sin 3t) + kA + kB \sin 8t = W + 8 \sin 8t
\]

\[
kA + (kB - 64mB) \sin 8t = W + 8 \sin 8t
\]

Matching the coefficients, we have

\[
kA = W
\]

\[
kB - 64mB = 8.
\]

Solving this system yields

\[
A = \frac{W}{k} \quad \text{and} \quad B = \frac{8}{k - 64m}.
\]

The particular solution is then

\[
x_p(t) = \frac{W}{k} + \frac{8}{k - 64m} \sin 8t,
\]

which means the general solution is

\[
x(t) = C_3 \cos \omega t + C_4 \sin \omega t + \frac{W}{k} + \frac{8}{k - 64m} \sin 8t
\]

\[
= C_3 \cos \sqrt{\frac{k}{m}} t + C_4 \sin \sqrt{\frac{k}{m}} t + \frac{W}{k} + \frac{8}{k - 64m} \sin 8t.
\]

www.stemjock.com
Take a derivative of it with respect to $t$.

$$x'(t) = -C_3 \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}} t + C_4 \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}} t + \frac{64}{k - 64m} \cos 8t$$

Now apply the initial conditions,

$$x(0) = 6 \text{ in} + 3 \text{ in} = 9 \text{ in} \times \frac{1 \text{ ft}}{12 \text{ in}} = \frac{3}{4} \text{ ft}$$

$$x'(0) = 0 \frac{\text{ft}}{\text{s}},$$

to determine $C_3$ and $C_4$.

$$x(0) = C_3 + \frac{W}{k} = \frac{3}{4}$$

$$x'(0) = C_4 \sqrt{\frac{k}{m}} + \frac{64}{k - 64m} = 0$$

Solving this system of equations yields

$$C_3 = \frac{3}{4} - \frac{W}{k} = \frac{1}{4} \quad \text{and} \quad C_4 = -\frac{64}{k - 64m} \sqrt{\frac{m}{k}},$$

so

$$x(t) = \frac{1}{4} \cos \sqrt{\frac{k}{m}} t - \frac{64}{k - 64m} \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}} t + \frac{W}{k} + \frac{8}{k - 64m} \sin 8t.$$  

Finally, plug in the numbers, $W = 8 \text{ lb}$ and $k = 16 \text{ lb/ft}$. The mass $m$ is obtained by dividing the weight by the gravity: $m = W/g = 8 \text{ lb}/(32.2 \text{ ft/s}^2)$. As $x(t)$ is in feet, multiply the result by 12 to convert it to inches.
Zoom in the graph to the first few oscillations. The points where the slope is zero are marked approximately.