Problem 12

A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3 \cos 3t - 2 \sin 3t) \text{ N}$, determine the steady state response. Express your answer in the form $R \cos(\omega t - \delta)$.

Solution

Start by drawing a free-body diagram of the mass.

![Free-body diagram of a spring-mass system](image)

Apply Newton’s second law in the $x$-direction to obtain the equation of motion for the mass.

$$
\sum F_x = ma_x
$$

$$
-x' - 3x + 2g + 3 \cos 3t - 2 \sin 3t = 2a_x
$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$
-x' - 3x + 2g + 3 \cos 3t - 2 \sin 3t = 2x''
$$

$$
2x'' + x' + 3x = 2g + 3 \cos 3t - 2 \sin 3t
$$

This is a linear inhomogeneous ODE, so its general solution can be expressed as the sum of the complementary solution and the particular solution.

$$
x(t) = x_c(t) + x_p(t)
$$

The complementary solution satisfies the associated homogeneous equation.

$$
2x''_c + x'_c + 3x_c = 0 \quad (1)
$$

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Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$x_c = e^{rt} \rightarrow x'_c = re^{rt} \rightarrow x''_c = r^2 e^{rt}$

Substitute these expressions to obtain an algebraic equation for $r$.

$$2(r^2 e^{rt}) + re^{rt} + 3(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$2r^2 + r + 3 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(2)(3)}}{2(2)} = \frac{-1 \pm \sqrt{-23}}{4} = \frac{-1 \pm i\sqrt{23}}{4} = -\frac{1}{4} \pm i\mu$$

Two solutions to equation (1) are then

$$x_c = e^{(-1/4-i\mu)t} \quad \text{and} \quad x_c = e^{(-1/4+i\mu)t}.$$  

By the principle of superposition, the general solution for $x_c$ is a linear combination of these two.

$$x_c(t) = C_1e^{(-1/4-i\mu)t} + C_2e^{(-1/4+i\mu)t}$$

$$= C_1e^{-t/4-i\mu t} + C_2e^{-t/4+i\mu t}$$

$$= C_1e^{-t/4}e^{-i\mu t} + C_2e^{-t/4}e^{i\mu t}$$

$$= C_1e^{-t/4}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-t/4}[\cos(\mu t) + i\sin(\mu t)]$$

$$= C_1e^{-t/4}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-t/4}[\cos(\mu t) + i\sin(\mu t)]$$

$$= C_1e^{-t/4}[\cos(\mu t) + iC_1e^{-t/4} \sin(\mu t)] + C_2e^{-t/4}[\cos(\mu t) - iC_2e^{-t/4} \sin(\mu t)]$$

$$= (C_1 + C_2)e^{-t/4} cos(\mu t) + (-iC_1 + iC_2)e^{-t/4} \sin(\mu t)$$

$$= C_3e^{-t/4} \cos(\mu t) + C_4e^{-t/4} \sin(\mu t)$$

$$= e^{-t/4}(C_3 \cos(\mu t) + C_4 \sin(\mu t))$$

On the other hand, the particular solution satisfies

$$2x_p'' + x'_p + 3x_p = 2g + 3 \cos 3t - 2 \sin 3t.$$  

For the first term on the right side, we'll include a constant $A$ in the trial solution. To account for the sine and cosine terms, we'll include $B \cos 3t + C \sin 3t$ in the trial solution. Substitute $x_p(t) = A + B \cos 3t + C \sin 3t$ into the equation to determine $A$ and $B$ and $C$.

$$2(A + B \cos 3t + C \sin 3t)'' + (A + B \cos 3t + C \sin 3t)' + 3(A + B \cos 3t + C \sin 3t) = 2g + 3 \cos 3t - 2 \sin 3t$$

$$2(-9B \cos 3t - 9C \sin 3t) + (-3B \sin 3t + 3C \cos 3t) + 3(A + B \cos 3t + C \sin 3t) = 2g + 3 \cos 3t - 2 \sin 3t$$

$$3A + (-18B + 3C + 3B) \cos 3t + (-18C - 3B + 3C) \sin 3t = 2g + 3 \cos 3t - 2 \sin 3t$$

Matching the coefficients, we have

$$3A = 2g$$
$$-18B + 3C + 3B = 3$$
$$-18C - 3B + 3C = -2.$$  

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Solving this system yields

\[ A = \frac{2g}{3} \quad \text{and} \quad B = -\frac{1}{6} \quad \text{and} \quad C = \frac{1}{6}. \]

The particular solution is then

\[ x_p(t) = \frac{2g}{3} - \frac{1}{6} \cos 3t + \frac{1}{6} \sin 3t, \]

which means the general solution is

\[ x(t) = e^{-t/4} \left( C_3 \cos \sqrt{\frac{23}{4}} t + C_4 \sin \sqrt{\frac{23}{4}} t \right) + \frac{2g}{3} - \frac{1}{6} \cos 3t + \frac{1}{6} \sin 3t. \]

To combine the two sinusoidal terms into one, introduce an amplitude \( R \) and a phase \( \delta \).

\[ x(t) = e^{-t/4} \left( C_3 \cos \sqrt{\frac{23}{4}} t + C_4 \sin \sqrt{\frac{23}{4}} t \right) + \frac{2g}{3} + R \cos \delta \cos 3t + R \sin \delta \sin 3t \]

\[ = e^{-t/4} \left( C_3 \cos \sqrt{\frac{23}{4}} t + C_4 \sin \sqrt{\frac{23}{4}} t \right) + \frac{2g}{3} + R \cos(3t - \delta) \]

\( R \) and \( \delta \) satisfy the following system of equations.

\[ R \cos \delta = -\frac{1}{6} \quad \text{(2)} \]

\[ R \sin \delta = \frac{1}{6} \quad \text{(3)} \]

To find \( R \), square both sides of each equation and then add the respective sides together.

\[ R = \sqrt{\left( -\frac{1}{6} \right)^2 + \left( \frac{1}{6} \right)^2} = \frac{1}{3\sqrt{2}} \]

To find \( \delta \), divide the respective sides of equation (3) by those of equation (2).

\[ \tan \delta = -1 \]

\[ \delta = \tan^{-1}(-1) = -\tan^{-1}1 + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} \]

Therefore,

\[ x(t) = e^{-t/4} \left( C_3 \cos \sqrt{\frac{23}{4}} t + C_4 \sin \sqrt{\frac{23}{4}} t \right) + \frac{2g}{3} + \frac{1}{3\sqrt{2}} \cos \left( 3t - \frac{3\pi}{4} \right). \]