Problem 6

A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of \(10 \sin(t/2)\) N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.

The gravitational and spring forces balance each other.

\[ mg = k\Delta x \]

From this equation, \( k \) can be determined.

\[
(5 \text{ kg}) \left(9.81 \frac{\text{m}}{\text{s}^2}\right) = k \left(10 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}}\right)
\]

\[ k = 490.5 \frac{\text{N}}{\text{m}} \]
Consider now the mass in nonequilibrium. Assume that the damping force $F_d$ is proportional to the speed of the mass.

$$ F_d \propto x' $$

Introduce a proportionality constant $c$ to change this to an equation.

$$ F_d = cx' $$

Given that the damping force is 2 N when the speed is 4 cm/s, $c$ can be determined.

$$ 2 \text{ N} = c \left(4 \frac{\text{cm}}{\text{s}} \times \frac{1 \text{ m}}{100 \text{ cm}}\right) $$

$$ c = 50 \frac{\text{N} \cdot \text{s}}{\text{m}} $$

Draw a free-body diagram for the mass in nonequilibrium.

![Free-body diagram](image)

Apply Newton’s second law in the $x$-direction to obtain the equation of motion for the mass.

$$ \sum F_x = ma_x $$

$$ -cx' - kx + mg + 10 \sin \frac{t}{2} = ma_x $$

Use the fact that acceleration is the second derivative of position with respect to time.

$$ -mx'' - kx + mg + 10 \sin \frac{t}{2} = mx'' $$

$$ mx'' + cx' + kx = mg + 10 \sin \frac{t}{2} $$

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Plug in the values for the constants.

\[ 5x'' + 50x' + 490.5x = 5g + 10 \sin \frac{t}{2} \]

Divide both sides by 5.

\[ x'' + 10x' + 98.1x = g + 2 \sin \frac{t}{2} \]

This is a linear inhomogeneous ODE, so its general solution can be expressed as sum of the complementary solution and the particular solution.

\[ x(t) = x_c(t) + x_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ x''_c + 10x'_c + 98.1x_c = 0 \] (1)

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form \( x_c = e^{rt} \).

\[ x_c = e^{rt} \quad \rightarrow \quad x'_c = re^{rt} \quad \rightarrow \quad x''_c = r^2e^{rt} \]

Substitute these expressions to obtain an algebraic equation for \( r \).

\[ r^2e^{rt} + 10(re^{rt}) + 98.1(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 10r + 98.1 = 0 \]

\[ r = \frac{-10 \pm \sqrt{100 - 4(98.1)}}{2} = \frac{-10 \pm \sqrt{-292.4}}{2} = -5 \pm i\mu \]

Two solutions to equation (1) are then

\[ x_c = e^{(-5-i\mu)t} \quad \text{and} \quad x_c = e^{(-5+i\mu)t}. \]

By the principle of superposition, the general solution for \( x_c \) is a linear combination of these two.

\[ x_c(t) = C_1e^{(-5-i\mu)t} + C_2e^{(-5+i\mu)t} \]

\[ = C_1e^{-5t-i\mu t} + C_2e^{-5t+i\mu t} \]

\[ = C_1e^{-5t}e^{-i\mu t} + C_2e^{-5t}e^{i\mu t} \]

\[ = C_1e^{-5t}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-5t}[\cos(\mu t) + i\sin(\mu t)] \]

\[ = C_1e^{-5t}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-5t}[\cos(\mu t) + i\sin(\mu t)] \]

\[ = C_1e^{-5t}\cos \mu t - iC_1e^{-5t}\sin \mu t + C_2e^{-5t}\cos \mu t + iC_2e^{-5t}\sin \mu t \]

\[ = (C_1 + C_2)e^{-5t}\cos \mu t + (-iC_1 + iC_2)e^{-5t}\sin \mu t \]

\[ = C_3e^{-5t}\cos \mu t + C_4e^{-5t}\sin \mu t \]

\[ = e^{-5t}(C_3 \cos \mu t + C_4 \sin \mu t) \]

On the other hand, the particular solution satisfies

\[ x''_p + 10x'_p + 98.1x_p = g + 2 \sin \frac{t}{2} \]

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For the first term on the right side, we’ll include a constant $A$ in the trial solution. Also, since there are even and odd derivatives on the left side, we’ll include $B\cos(t/2) + C\sin(t/2)$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\cos(t/2) + C\sin(t/2)$ into the equation to determine $A$ and $B$ and $C$.

\[
\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right)'' + 10\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right)' + 98.1\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right) = g + 2\sin\frac{t}{2}
\]

\[
\left(-\frac{B}{4}\cos\frac{t}{2} - \frac{C}{4}\sin\frac{t}{2}\right) + 10\left(-\frac{B}{2}\sin\frac{t}{2} + \frac{C}{2}\cos\frac{t}{2}\right) + 98.1\left(A + B\cos\frac{t}{2} + C\sin\frac{t}{2}\right) = g + 2\sin\frac{t}{2}
\]

\[
98.1A + (97.85B + 5C)\cos\frac{t}{2} + (-5B + 97.85C)\sin\frac{t}{2} = g + 2\sin\frac{t}{2}
\]

Matching the coefficients, we have

\[
98.1A = g
\]

\[
97.85B + 5C = 0
\]

\[
-5B + 97.85C = 2.
\]

Solving this system yields

\[
A = \frac{1}{10} = 0.1 \quad \text{and} \quad B = -\frac{4000}{3\,839\,849} \quad \text{and} \quad C = \frac{78\,280}{3\,839\,849}.
\]

The particular solution is then

\[
x_p(t) = \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2},
\]

which means the general solution is

\[
x(t) = e^{-5t}(C_3\cos\mu t + C_4\sin\mu t) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}
\]

\[
= e^{-5t}\left(C_3\cos\frac{\sqrt{292.4}}{2}t + C_4\sin\frac{\sqrt{292.4}}{2}t\right) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}
\]

\[
= e^{-5t}(C_3\cos\sqrt{73.1}t + C_4\sin\sqrt{73.1}t) + \frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}.
\]

Take a derivative of it with respect to $t$.

\[
x'(t) = -5e^{-5t}(C_3\cos\sqrt{73.1}t + C_4\sin\sqrt{73.1}t) + e^{-5t}(-\sqrt{73.1}C_3\sin\sqrt{73.1}t + C_4\sqrt{73.1}\cos\sqrt{73.1}t)
\]

\[
+ \frac{2000}{3\,839\,849}\sin\frac{t}{2} + \frac{39\,140}{3\,839\,849}\cos\frac{t}{2}
\]

Now apply the initial conditions,

\[
x(0) = 10\text{\,cm} \times \frac{1\text{\,m}}{100\text{\,cm}} = \frac{1}{10}\text{\,m}
\]

\[
x'(0) = 3\text{\,cm/s} \times \frac{1\text{\,m}}{100\text{\,cm}} = \frac{3}{10}\text{\,m/s},
\]

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to determine $C_3$ and $C_4$.

\[
\begin{align*}
  x(0) &= C_3 + \frac{1}{10} - \frac{4000}{3839849} = \frac{1}{10} \\
  x'(0) &= -5C_3 + C_4\sqrt{73.1} + \frac{39140}{3839849} = \frac{3}{100}
\end{align*}
\]

Solving this system of equations yields

\[
C_3 = \frac{4000}{3839849} \quad \text{and} \quad C_4 = \frac{9605547}{383984900\sqrt{73.1}},
\]

so

\[
x(t) = e^{-5t} \left( \frac{4000}{3839849} \cos \sqrt{73.1}t + \frac{9605547}{383984900\sqrt{73.1}} \sin \sqrt{73.1}t \right) + \frac{1}{10} - \frac{4000}{3839849} \cos \frac{t}{2} + \frac{78280}{3839849} \sin \frac{t}{2}.
\]

As $x(t)$ is in meters, multiply the result by 100 to convert it to centimeters.