Problem 22

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

$$y^{(4)} + y = 0$$

Solution

Because this is a linear fourth-order ODE, there will be four solutions for it. Let y_1, y_2, y_3, and y_4 represent them and let $W = W(y_1, y_2, y_3, y_4)$ be the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Differentiate both sides with respect to t.

$$W' = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1''' & y_2''' & y_3''' & y_4''' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0}$$

$$+ \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix}$$

Since y_1, y_2, y_3, and y_4 are solutions to the ODE, they satisfy

$$\begin{aligned} y_1^{(4)} + y_1 = 0 \quad &\rightarrow \quad y_1^{(4)} = -y_1 \\ y_2^{(4)} + y_2 = 0 \quad &\rightarrow \quad y_2^{(4)} = -y_2 \\ y_3^{(4)} + y_3 = 0 \quad &\rightarrow \quad y_3^{(4)} = -y_3 \\ y_4^{(4)} + y_4 = 0 \quad &\rightarrow \quad y_4^{(4)} = -y_4. \end{aligned}$$

Substitute these formulas for y_1''', y_2''', y_3''', and $y^{(4)}$ into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ -y_1 & -y_2 & -y_3 & -y_4 \end{vmatrix}$$

Add the first row to the fourth row.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Therefore, integrating both sides with respect to t,

$$W(t) = c.$$