Problem 12

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y''' - 3y'' + 3y' - y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - 3(r^2 e^{rt}) + 3(re^{rt}) - (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

$$r = \{1\}$$

One solution to the ODE is then $y = e^t$. By using the method of reduction of order, we can obtain the general solution. Plug in $y(t) = c(t)e^t$ to the ODE.

$$[c(t)e^t]''' - 3[c(t)e^t]'' + 3[c(t)e^t]' - [c(t)e^t] = 0$$

Evaluate the derivatives.

$$[c'(t)e^t + c(t)e^t]'' - 3[c'(t)e^t + c(t)e^t]' + 3[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 0$$

$$[c''(t)e^t + 2c'(t)e^t + c(t)e^t]' - 3[c''(t)e^t + 2c'(t)e^t + c(t)e^t] + 3[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 0$$

$$[c'''(t)e^t + 3c''(t)e^t + 3c'(t)e^t + c(t)e^t] - 3[c''(t)e^t + 2c'(t)e^t + c(t)e^t] + 3[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 0$$

Expand the left side.

$$c'''(t)e^t + \cancel{3c''(t)e^t} + \cancel{3c'(t)e^t} + \cancel{c(t)e^t} - \cancel{3c''(t)e^t} - \cancel{6c'(t)e^t} - \cancel{3c(t)e^t} + \cancel{3c'(t)e^t} + \cancel{3c(t)e^t} - \cancel{c(t)e^t} = 0$$

$$c'''(t)e^t = 0$$

Divide both sides by e^t.

$$c'''(t) = 0$$

Integrate both sides with respect to t.

$$c''(t) = C_1$$

Integrate both sides with respect to t again.

$$c'(t) = C_1 t + C_2$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{C_1}{2}t^2 + C_2 t + C_3$$

Therefore, since $y(t) = c(t)e^t$,

$$y(t) = C_4 t^2 e^t + C_2 te^t + C_3 e^t,$$

where a new constant C_4 is used for $C_1/2$.