Problem 21

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(8)} + 8y^{(4)} + 16y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y^{(4)} = r^4 e^{rt} \quad \rightarrow \quad y^{(8)} = r^8 e^{rt}$$

Substitute these expressions into the ODE.

$$r^8 e^{rt} + 8(r^4 e^{rt}) + 16(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^8 + 8r^4 + 16 = 0$$
$$(r^4 + 4)^2 = 0 \tag{1}$$
$$r^4 + 4 = 0$$
$$r^4 = -4$$
$$r = (-4)^{1/4}$$
$$r = (4e^{i\pi})^{1/4}$$
$$r = [4e^{i(\pi + 2n\pi)}]^{1/4}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$r = 4^{1/4} e^{i(\pi/4 + n\pi/2)}$$

The four distinct roots are obtained by setting $n = 0$, $n = 1$, $n = 2$, and $n = 3$. Other values of n lead to redundant roots.

$$n = 0: \quad r = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$$
$$n = 1: \quad r = \sqrt{2}e^{3i\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -1 + i$$
$$n = 2: \quad r = \sqrt{2}e^{5i\pi/4} = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -1 - i$$
$$n = 3: \quad r = \sqrt{2}e^{7i\pi/4} = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 1 - i$$

Four solutions to the ODE are then $y = e^{(1+i)t}$ and $y = e^{(-1+i)t}$ and $y = e^{(-1-i)t}$ and $y = e^{(1-i)t}$. The multiplicity of each of these roots is 2 because the quantity in equation (1) is squared. That means a second linearly independent solution can be obtained from each of these four by including a factor of t: $y = te^{(1+i)t}$ and $y = te^{(-1+i)t}$ and $y = te^{(-1-i)t}$ and $y = te^{(1-i)t}$. By the principle of superposition, the general solution for y is a linear combination of these eight.

$$y(t) = C_1 e^{(1+i)t} + C_2 e^{(-1+i)t} + C_3 e^{(-1-i)t} + C_4 e^{(1-i)t} + C_5 te^{(1+i)t} + C_6 te^{(-1+i)t} + C_7 te^{(-1-i)t} + C_8 te^{(1-i)t}$$

Use Euler's formula to write $y(t)$ in terms of real functions.

$$\begin{aligned}
y(t) &= C_1e^{t+it} + C_2e^{-t+it} + C_3e^{-t-it} + C_4e^{t-it} + C_5te^{t+it} + C_6te^{-t+it} + C_7te^{-t-it} + C_8te^{t-it} \\
&= C_1e^te^{it} + C_2e^{-t}e^{it} + C_3e^{-t}e^{-it} + C_4e^te^{-it} + C_5te^te^{it} + C_6te^{-t}e^{it} + C_7te^{-t}e^{-it} + C_8te^te^{-it} \\
&= e^t(C_1e^{it} + C_4e^{-it} + C_5te^{it} + C_8te^{-it}) + e^{-t}(C_2e^{it} + C_3e^{-it} + C_6te^{it} + C_7te^{-it}) \\
&= e^t[C_1(\cos t + i\sin t) + C_4(\cos t - i\sin t) + C_5t(\cos t + i\sin t) + C_8t(\cos t - i\sin t)] \\
&\qquad + e^{-t}[C_2(\cos t + i\sin t) + C_3(\cos t - i\sin t) + C_6t(\cos t + i\sin t) + C_7t(\cos t - i\sin t)] \\
&= e^t[(C_1 + C_4)\cos t + (C_5 + C_8)t\cos t + (iC_1 - iC_4)\sin t + (iC_5 - iC_8)t\sin t] \\
&\qquad + e^{-t}[(C_2 + C_3)\cos t + (C_6 + C_7)t\cos t + (iC_2 - iC_3)\sin t + (iC_6 - iC_7)t\sin t]
\end{aligned}$$

Therefore, using new arbitrary constants,

$$\begin{aligned}
y(t) = e^t(C_9\cos t + C_{10}t\cos t + C_{11}\sin t + C_{12}t\sin t) \\
+ e^{-t}(C_{13}\cos t + C_{14}t\cos t + C_{15}\sin t + C_{16}t\sin t).
\end{aligned}$$