Problem 23

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y''' - 5y'' + 3y' + y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - 5(r^2 e^{rt}) + 3(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - 5r^2 + 3r + 1 = 0$$

$$(r-1)(r^2 - 4r - 1) = 0$$

Use the zero product theorem.

$$r - 1 = 0 \quad \text{or} \quad r^2 - 4r - 1 = 0$$

$$r = 1 \quad \text{or} \quad r = \frac{4 \pm \sqrt{16 - 4(-1)(1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

$$r = \{2 - \sqrt{5}, 1, 2 + \sqrt{5}\}$$

Three solutions to the ODE are then $y = e^{(2-\sqrt{5})t}$ and $y = e^t$ and $y = e^{(2+\sqrt{5})t}$. By the principle of superposition, the general solution for y is a linear combination of these three.

$$y(t) = C_1 e^{(2-\sqrt{5})t} + C_2 e^t + C_3 e^{(2+\sqrt{5})t}$$