Problem 24

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y''' + 5y'' + 6y' + 2y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} + 5(r^2 e^{rt}) + 6(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 + 5r^2 + 6r + 2 = 0$$

$$(r+1)(r^2 + 4r + 2) = 0$$

Use the zero product theorem.

$$\begin{aligned} r + 1 = 0 \quad &\text{or} \quad r^2 + 4r + 2 = 0 \\ r = -1 \quad &\text{or} \quad r = \frac{-4 \pm \sqrt{16 - 4(2)(1)}}{2} = \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2} \end{aligned}$$

$$r = \{-2 - \sqrt{2}, -1, -2 + \sqrt{2}\}$$

Three solutions to the ODE are then $y = e^{(-2-\sqrt{2})t}$ and $y = e^{-t}$ and $y = e^{(-2+\sqrt{2})t}$. By the principle of superposition, the general solution for y is a linear combination of these three.

$$y(t) = C_1 e^{(-2-\sqrt{2})t} + C_2 e^{-t} + C_3 e^{(-2+\sqrt{2})t}$$