Problem 32

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y''' - y'' + y' - y = 0; \qquad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2e^{rt} \quad \to \quad y''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - r^2e^{rt} + re^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r^2 + r - 1 = 0$$
$$(r-1)(r^2+1) = 0$$
$$r = \{1, -i, i\}$$

Three solutions to the ODE are then $y = e^t$ and $y = e^{-it}$ and $y = e^{it}$. By the principle of superposition, the general solution for y is a linear combination of these three.

$$\begin{aligned} y(t) &= C_1e^t + C_2e^{-it} + C_3e^{it} \\ &= C_1e^t + C_2(\cos t - i\sin t) + C_3(\cos t + i\sin t) \\ &= C_1e^t + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\ &= C_1e^t + C_4\cos t + C_5\sin t \end{aligned}$$

Differentiate this solution twice with respect to t.

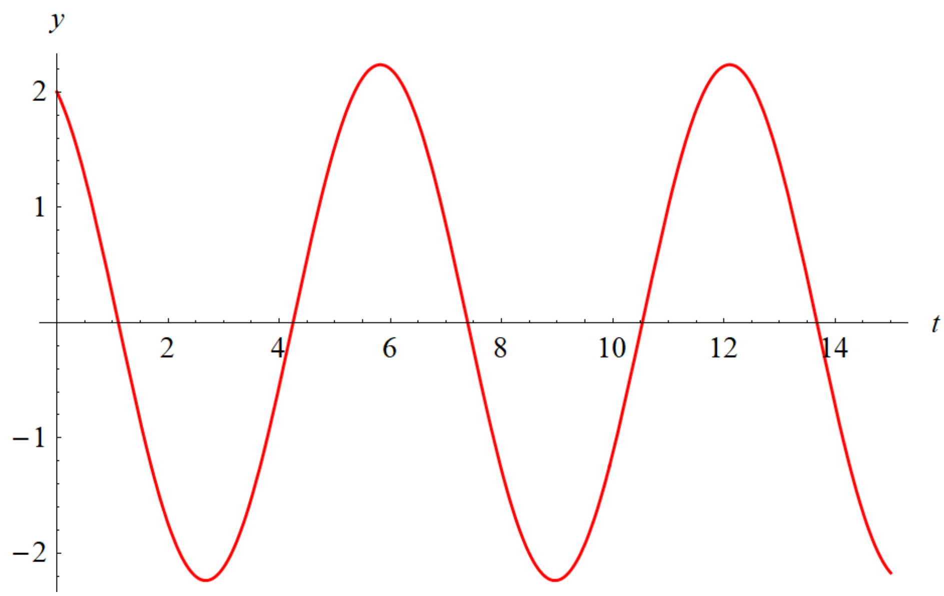
$$\begin{aligned} y'(t) &= C_1e^t - C_4\sin t + C_5\cos t \\ y''(t) &= C_1e^t - C_4\cos t - C_5\sin t \end{aligned}$$

Apply the initial conditions now to determine C_1, C_4, and C_5.

$$\begin{aligned} y(0) &= C_1 + C_4 = 2 \\ y'(0) &= C_1 + C_5 = -1 \\ y''(0) &= C_1 - C_4 = -2 \end{aligned}$$

Solving this system of equations yields $C_1 = 0$, $C_4 = 2$, and $C_5 = -1$. Therefore,

$$y(t) = 2\cos t - \sin t.$$



The solution oscillates like this for all t.