Problem 33

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0; \qquad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt} \quad \to \quad y^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^4 e^{rt}) - r^3 e^{rt} - 9(r^2 e^{rt}) + 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$2r^4 - r^3 - 9r^2 + 4r + 4 = 0$$

$$(r+2)(2r+1)(r-1)(r-2) = 0$$

$$r = \left\{-2, -\frac{1}{2}, 1, 2\right\}$$

Four solutions to the ODE are then $y = e^{-2t}$ and $y = e^{-t/2}$ and $y = e^t$ and $y = e^{2t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$y(t) = C_1 e^{-2t} + C_2 e^{-t/2} + C_3 e^t + C_4 e^{2t}$$

Differentiate this solution three times with respect to t.

$$y'(t) = -2C_1 e^{-2t} - \frac{C_2}{2} e^{-t/2} + C_3 e^t + 2C_4 e^{2t}$$

$$y''(t) = 4C_1 e^{-2t} + \frac{C_2}{4} e^{-t/2} + C_3 e^t + 4C_4 e^{2t}$$

$$y'''(t) = -8C_1 e^{-2t} - \frac{C_2}{8} e^{-t/2} + C_3 e^t + 8C_4 e^{2t}$$

Apply the initial conditions now to determine C_1, C_2, C_3, and C_4.

$$y(0) = C_1 + C_2 + C_3 + C_4 = -2$$

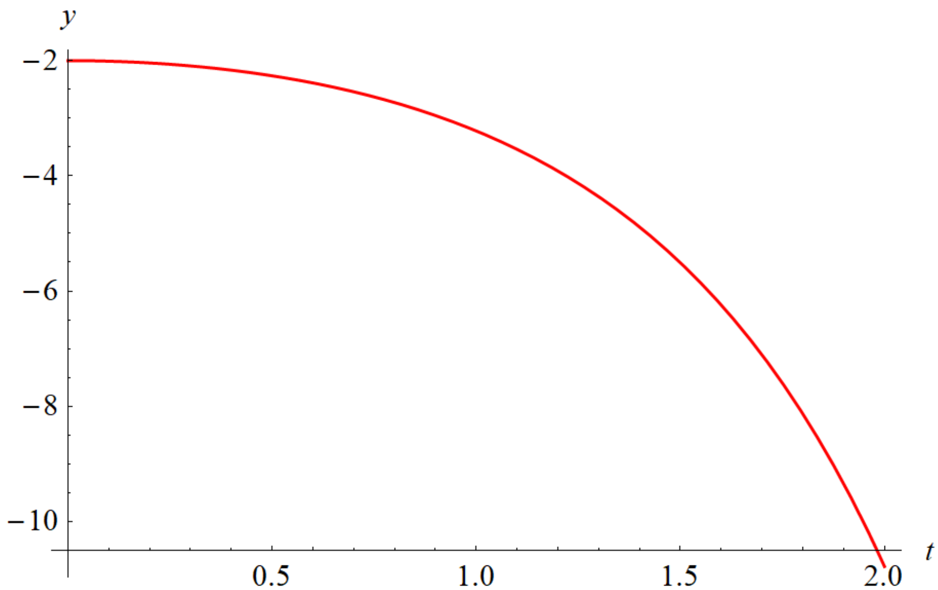
$$y'(0) = -2C_1 - \frac{C_2}{2} + C_3 + 2C_4 = 0$$

$$y''(0) = 4C_1 + \frac{C_2}{4} + C_3 + 4C_4 = -2$$

$$y'''(0) = -8C_1 - \frac{C_2}{8} + C_3 + 8C_4 = 0$$

Solving this system of equations yields $C_1 = -1/6$, $C_2 = -16/15$, $C_3 = -2/3$, and $C_4 = -1/10$. Therefore,

$$y(t) = -\frac{1}{6} e^{-2t} - \frac{16}{15} e^{-t/2} - \frac{2}{3} e^t - \frac{1}{10} e^{2t}.$$



The first two terms tend to zero in the limit as $t \to \infty$, but the other two terms blow up and make $y(t) \to -\infty$ in the limit.