Problem 34

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$4y''' + y' + 5y = 0; \qquad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2e^{rt} \quad \to \quad y''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^3e^{rt}) + re^{rt} + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$4r^3 + r + 5 = 0$$

$$(r+1)(4r^2 - 4r + 5) = 0$$

Use the zero product theorem.

$$\begin{aligned} r + 1 &= 0 \quad \text{or} \quad 4r^2 - 4r + 5 = 0 \\ r &= -1 \quad \text{or} \quad r = \frac{4 \pm \sqrt{16 - 4(4)(5)}}{2(4)} = \frac{4 \pm \sqrt{-64}}{8} = \frac{4 \pm 8i}{8} = \frac{1}{2} \pm i \end{aligned}$$

$$r = \left\{-1, \frac{1}{2} - i, \frac{1}{2} + i\right\}$$

Three solutions to the ODE are then $y = e^{-t}$ and $y = e^{(1/2-i)t}$ and $y = e^{(1/2+i)t}$. By the principle of superposition, the general solution is a linear combination of these three.

$$y(t) = C_1e^{-t} + C_2e^{(1/2-i)t} + C_3e^{(1/2+i)t}$$

Differentiate this solution twice with respect to t.

$$y'(t) = -C_1e^{-t} + C_2\left(\frac{1}{2} - i\right)e^{(1/2-i)t} + C_3\left(\frac{1}{2} + i\right)e^{(1/2+i)t}$$

$$y''(t) = C_1e^{-t} + C_2\left(\frac{1}{2} - i\right)^2e^{(1/2-i)t} + C_3\left(\frac{1}{2} + i\right)^2e^{(1/2+i)t}$$

Apply the initial conditions now to determine C_1, C_2, and C_3.

$$y(0) = C_1 + C_2 + C_3 = 2$$

$$y'(0) = -C_1 + C_2\left(\frac{1}{2} - i\right) + C_3\left(\frac{1}{2} + i\right) = 1$$

$$y''(0) = C_1 + C_2\left(\frac{1}{2} - i\right)^2 + C_3\left(\frac{1}{2} + i\right)^2 = -1$$

Solving this system of equations yields

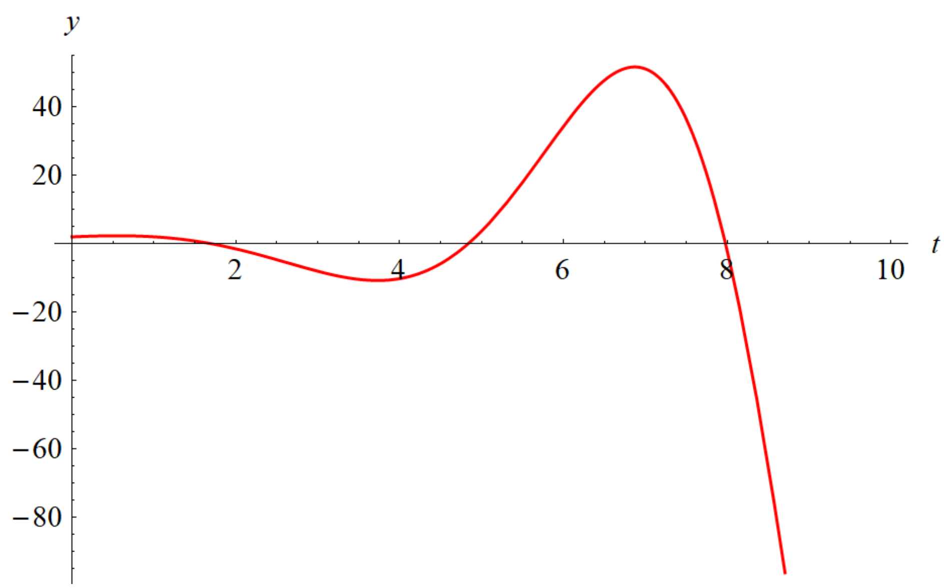
$$C_1 = \frac{2}{13} \quad \text{and} \quad C_2 = \frac{12}{13} + i\frac{3}{26} \quad \text{and} \quad C_3 = \frac{12}{13} - i\frac{3}{26}.$$

Now that the constants are known, write $y(t)$ in terms of real functions.

$$\begin{aligned}
y(t) &= C_1e^{-t} + C_2e^{t/2-it} + C_3e^{t/2+it} \\
&= C_1e^{-t} + C_2e^{t/2}e^{-it} + C_3e^{t/2}e^{it} \\
&= C_1e^{-t} + e^{t/2}(C_2e^{-it} + C_3e^{it}) \\
&= C_1e^{-t} + e^{t/2}[C_2(\cos t - i\sin t) + C_3(\cos t + i\sin t)] \\
&= C_1e^{-t} + e^{t/2}[(C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t]
\end{aligned}$$

Therefore,

$$y(t) = \frac{2}{13}e^{-t} + e^{t/2}\left(\frac{24}{13}\cos t + \frac{3}{13}\sin t\right).$$



The first term tends to zero in the limit as $t \to \infty$, but the second one blows up and makes $y(t)$ diverge.