Problem 35

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$6y''' + 5y'' + y' = 0; \qquad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$6(r^3 e^{rt}) + 5(r^2 e^{rt}) + re^{rt} = 0$$

Divide both sides by e^{rt}.

$$6r^3 + 5r^2 + r = 0$$
$$r(2r+1)(3r+1) = 0$$
$$r = \left\{-\frac{1}{2}, -\frac{1}{3}, 0\right\}$$

Three solutions to the ODE are then $y = e^{-t/2}$ and $y = e^{-t/3}$ and $e^0 = 1$. By the principle of superposition, the general solution is a linear combination of these three.

$$y(t) = C_1 e^{-t/2} + C_2 e^{-t/3} + C_3$$

Differentiate it twice with respect to t.

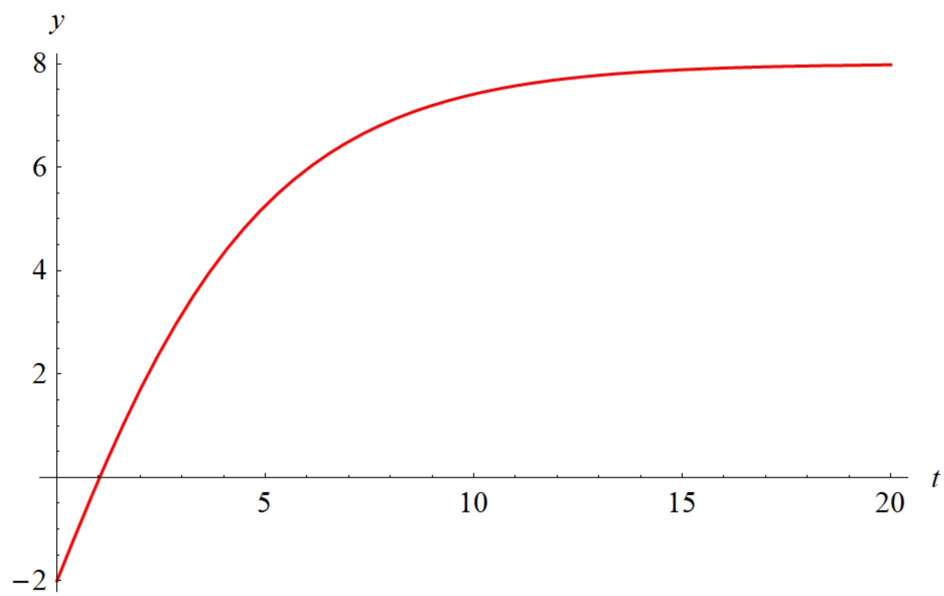
$$y'(t) = -\frac{C_1}{2} e^{-t/2} - \frac{C_2}{3} e^{-t/3}$$
$$y''(t) = \frac{C_1}{4} e^{-t/2} + \frac{C_2}{9} e^{-t/3}$$

Apply the initial conditions now to determine C_1, C_2, and C_3.

$$y(0) = C_1 + C_2 + C_3 = -2$$
$$y'(0) = -\frac{C_1}{2} - \frac{C_2}{3} = 2$$
$$y''(0) = \frac{C_1}{4} + \frac{C_2}{9} = 0$$

Solving this system of equations yields $C_1 = 8$, $C_2 = -18$, and $C_3 = 8$. Therefore,

$$y(t) = 8e^{-t/2} - 18e^{-t/3} + 8.$$



$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}(\underbrace{8e^{-t/2}}_{=\,0} - \underbrace{18e^{-t/3}}_{=\,0} + 8) \\ &= 8\end{aligned}$$