Problem 38

Consider the equation $y^{(4)} - y = 0$.

(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.

(b) Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.

(c) Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

Solution

Part (a)

Because this is a linear fourth-order ODE, there will be four solutions for it. Let y_1, y_2, y_3, and y_4 represent them and let $W = W(y_1, y_2, y_3, y_4)$ be the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Differentiate both sides with respect to t.

$$W' = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1''' & y_2''' & y_3''' & y_4''' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0}$$

$$+ \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix}$$

Since y_1, y_2, y_3, and y_4 are solutions to the ODE, they satisfy

$$\begin{aligned} y_1^{(4)} - y_1 = 0 \quad &\rightarrow \quad y_1^{(4)} = y_1 \\ y_2^{(4)} - y_2 = 0 \quad &\rightarrow \quad y_2^{(4)} = y_2 \\ y_3^{(4)} - y_3 = 0 \quad &\rightarrow \quad y_3^{(4)} = y_3 \\ y_4^{(4)} - y_4 = 0 \quad &\rightarrow \quad y_4^{(4)} = y_4. \end{aligned}$$

Substitute these formulas for y_1''', y_2''', y_3''', and $y^{(4)}$ into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$$

Multiply the first row by -1 and add it to the fourth row. Doing so makes each entry in the bottom row zero.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Consequently, the determinant is zero.

$$W' = 0$$

Integrate both sides with respect to t

$$W(t) = c$$

Part (b)

Now calculate the Wronskian of e^t, e^{-t}, $\cos t$, and $\sin t$.

$$\begin{aligned}
W(e^t, e^{-t}, \cos t, \sin t) &= \begin{vmatrix} e^t & e^{-t} & \cos t & \sin t \\ (e^t)' & (e^{-t})' & (\cos t)' & (\sin t)' \\ (e^t)'' & (e^{-t})'' & (\cos t)'' & (\sin t)'' \\ (e^t)''' & (e^{-t})''' & (\cos t)''' & (\sin t)''' \end{vmatrix} \\
&= \begin{vmatrix} e^t & e^{-t} & \cos t & \sin t \\ e^t & -e^{-t} & -\sin t & \cos t \\ e^t & e^{-t} & -\cos t & -\sin t \\ e^t & -e^{-t} & \sin t & -\cos t \end{vmatrix} \\
&= e^t \begin{vmatrix} -e^{-t} & -\sin t & \cos t \\ e^{-t} & -\cos t & -\sin t \\ -e^{-t} & \sin t & -\cos t \end{vmatrix} - e^t \begin{vmatrix} e^{-t} & \cos t & \sin t \\ e^{-t} & -\cos t & -\sin t \\ -e^{-t} & \sin t & -\cos t \end{vmatrix} \\
&\qquad + e^t \begin{vmatrix} e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \\ -e^{-t} & \sin t & -\cos t \end{vmatrix} - e^t \begin{vmatrix} e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \\ e^{-t} & -\cos t & -\sin t \end{vmatrix} \\
&= e^t \left[-e^{-t} \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - e^{-t} \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} - e^{-t} \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \right] \\
&\quad - e^t \left[e^{-t} \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - e^{-t} \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} - e^{-t} \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} \right] \\
&\qquad + e^t \left[e^{-t} \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} + e^{-t} \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} - e^{-t} \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \right] \\
&\qquad\quad - e^t \left[e^{-t} \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} + e^{-t} \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} + e^{-t} \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \right] \\
&= e^t[-e^{-t}(\cos^2 t + \sin^2 t) - e^{-t}(0) - e^{-t}(\sin^2 t + \cos^2 t)] \\
&\quad - e^t[e^{-t}(\cos^2 t + \sin^2 t) - e^{-t}(-\cos^2 t - \sin^2 t) - e^{-t}(0)] \\
&\qquad + e^t[e^{-t}(0) + e^{-t}(-\cos^2 t - \sin^2 t) - e^{-t}(\cos^2 t + \sin^2 t)] \\
&\qquad\quad - e^t[e^{-t}(\sin^2 t + \cos^2 t) + e^{-t}(0) + e^{-t}(\cos^2 t + \sin^2 t)] \\
&= e^t(-2e^{-t}) - e^t(2e^{-t}) + e^t(-2e^{-t}) - e^t(2e^{-t}) \\
&= -2 - 2 - 2 - 2 = -8
\end{aligned}$$

Part (c)

Now calculate the Wronskian of $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

$$
\begin{aligned}
W(\cosh t, \sinh t, \cos t, \sin t) &= \begin{vmatrix} \cosh t & e^{-t} & \cos t & \sin t \\ (\cosh t)' & (\sinh t)' & (\cos t)' & (\sin t)' \\ (\cosh t)'' & (\sinh t)'' & (\cos t)'' & (\sin t)'' \\ (\cosh t)''' & (\sinh t)''' & (\cos t)''' & (\sin t)''' \end{vmatrix} \\
&= \begin{vmatrix} \cosh t & \sinh t & \cos t & \sin t \\ \sinh t & \cosh t & -\sin t & \cos t \\ \cosh t & \sinh t & -\cos t & -\sin t \\ \sinh t & \cosh t & \sin t & -\cos t \end{vmatrix} \\
&= \cosh t \begin{vmatrix} \cosh t & -\sin t & \cos t \\ \sinh t & -\cos t & -\sin t \\ \cosh t & \sin t & -\cos t \end{vmatrix} - \sinh t \begin{vmatrix} \sinh t & \cos t & \sin t \\ \sinh t & -\cos t & -\sin t \\ \cosh t & \sin t & -\cos t \end{vmatrix} \\
&\qquad + \cosh t \begin{vmatrix} \sinh t & \cos t & \sin t \\ \cosh t & -\sin t & \cos t \\ \cosh t & \sin t & -\cos t \end{vmatrix} - \sinh t \begin{vmatrix} \sinh t & \cos t & \sin t \\ \cosh t & -\sin t & \cos t \\ \sinh t & -\cos t & -\sin t \end{vmatrix} \\
&= \cosh t \left[\cosh t \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - \sinh t \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} + \cosh t \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}\right] \\
&\quad - \sinh t \left[\sinh t \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - \sinh t \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} + \cosh t \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix}\right] \\
&\quad + \cosh t \left[\sinh t \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} - \cosh t \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} + \cosh t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\right] \\
&\quad - \sinh t \left[\sinh t \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} - \cosh t \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} + \sinh t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\right] \\
&= \cosh t[\cosh t(\cos^2 t + \sin^2 t) - \sinh t(0) + \cosh t(\sin^2 t + \cos^2 t)] \\
&\quad - \sinh t[\sinh t(\cos^2 t + \sin^2 t) - \sinh t(-\cos^2 t - \sin^2 t) + \cosh t(0)] \\
&\quad + \cosh t[\sinh t(0) - \cosh t(-\cos^2 t - \sin^2 t) + \cosh t(\cos^2 t + \sin^2 t)] \\
&\quad - \sinh t[\sinh t(\sin^2 t + \cos^2 t) - \cosh t(0) + \sinh t(\cos^2 t + \sin^2 t)] \\
&= \cosh t(2\cosh t) - \sinh t(2\sinh t) + \cosh t(2\cosh t) - \sinh t(2\sinh t) \\
&= 4\cosh^2 t - 4\sinh^2 t \\
&= 4(\cosh^2 t - \sinh^2 t) \\
&= 4
\end{aligned}
$$