Problem 39

Consider the spring-mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.

(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \tag{i}$$

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0. \tag{ii}$$

Find the general solution of Eq. (ii).

(c) Suppose that the initial conditions are

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0. \tag{iii}$$

Use the first of Eqs. (i) and the initial conditions (iii) to obtain values for $u_1''(0)$ and $u_1'''(0)$. Then show that the solution of Eq. (ii) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2\cos t$.

(d) Now suppose that the initial conditions are

$$u_1(0) = -2, \qquad u_1'(0) = 0, \qquad u_2(0) = 1, \qquad u_2'(0) = 0. \tag{iv}$$

Proceed as in part (c) to show that the corresponding solutions are $u_1(t) = -2\cos\sqrt{6}t$ and $u_2(t) = \cos\sqrt{6}t$.

(e) Observe that the solutions obtained in parts (c) and (d) describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together; the second mass moves twice as far as the first. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up, and vice versa. In this mode the first mass moves twice as far as the second. For other initial conditions, not proportional to either of Eqs. (iii) or (iv), the motion of the masses is a combination of these two modes.

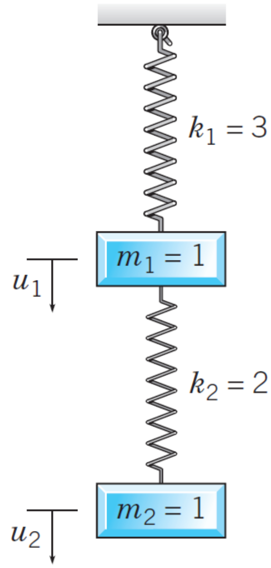
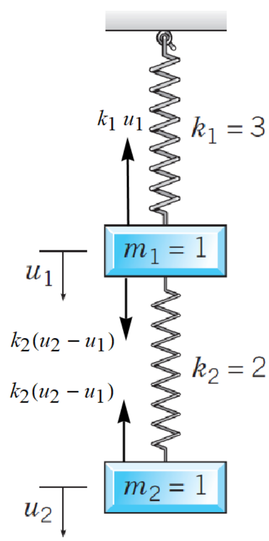


FIGURE 4.2.4 A two-spring, two-mass system.

Solution

Part (a)

Start by drawing free-body diagrams for the two masses. If m_1 is displaced a distance u_1 and m_2 is displaced a distance u_2, then we have the following two diagrams.



Apply Newton's second law twice—once for m_1 and once for m_2.

$$k_2(u_2 - u_1) - k_1 u_1 = m_1 a_1$$
$$-k_2(u_2 - u_1) = m_2 a_2$$

Plug in the numbers for k_1, k_2, m_1, and m_2 and use the fact that acceleration is the second derivative of position.

$$\begin{aligned} 2(u_2 - u_1) - 3u_1 &= u_1'' \\ -2(u_2 - u_1) &= u_2'' \end{aligned}$$

Therefore, the equations of motion are

$$u_1'' + 5u_1 = 2u_2 \tag{1}$$
$$u_2'' + 2u_2 = 2u_1. \tag{2}$$

Part (b)

Divide both sides of equation (1) by 2

$$\frac{1}{2}u_1'' + \frac{5}{2}u_1 = u_2$$

and plug this formula for u_2 into equation (2).

$$\left(\frac{1}{2}u_1'' + \frac{5}{2}u_1\right)'' + 2\left(\frac{1}{2}u_1'' + \frac{5}{2}u_1\right) = 2u_1$$

$$\frac{1}{2}u_1^{(4)} + \frac{5}{2}u_1'' + u_1'' + 5u_1 = 2u_1$$

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0 \tag{3}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u_1 = e^{rt}$.

$$u_1 = e^{rt} \quad \to \quad u_1' = re^{rt} \quad \to \quad u_1'' = r^2 e^{rt} \quad \to \quad u_1''' = r^3 e^{rt} \quad \to \quad u_1^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} + 7(r^2 e^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 7r^2 + 6 = 0$$

$$(r^2 + 6)(r^2 + 1) = 0$$

$$r = \{-\sqrt{6}i, \sqrt{6}i, -i, i\}$$

Four solutions to equation (3) are then $u_1 = e^{-\sqrt{6}it}$ and $u_1 = e^{\sqrt{6}it}$ and $u_1 = e^{-it}$ and $u_1 = e^{it}$. By the principle of superposition, the general solution for u_1 is a linear combination of these four.

$$\begin{aligned} u_1(t) &= C_1 e^{-\sqrt{6}it} + C_2 e^{\sqrt{6}it} + C_3 e^{-it} + C_4 e^{it} \\ &= C_1(\cos\sqrt{6}t - i\sin\sqrt{6}t) + C_2(\cos\sqrt{6}t + i\sin\sqrt{6}t) + C_3(\cos t - i\sin t) + C_4(\cos t + i\sin t) \\ &= (C_1 + C_2)\cos\sqrt{6}t + (-iC_1 + iC_2)\sin\sqrt{6}t + (C_3 + C_4)\cos t + (-iC_3 + iC_4)\sin t \\ &= C_5\cos\sqrt{6}t + C_6\sin\sqrt{6}t + C_7\cos t + C_8\sin t \end{aligned}$$

Part (c)

Suppose first that the initial conditions are

$$\begin{aligned} u_1(0) &= 1 & u_2(0) &= 2 \\ u_1'(0) &= 0 & u_2'(0) &= 0. \end{aligned}$$

From equation (1), $u_1''(0)$ and $u_1'''(0)$ can be determined.

$$u_1'' + 5u_1 = 2u_2 \quad \rightarrow \quad u_1'' = 2u_2 - 5u_1 \quad \rightarrow \quad \begin{cases} u_1''(0) = 2u_2(0) - 5u_1(0) = -1 \\ u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 0 \end{cases}$$

Differentiate the general solution three times.

$$\begin{aligned} u_1'(t) &= -C_5\sqrt{6}\sin\sqrt{6}t + C_6\sqrt{6}\cos\sqrt{6}t - C_7\sin t + C_8\cos t \\ u_1''(t) &= -6C_5\cos\sqrt{6}t - 6C_6\sin\sqrt{6}t - C_7\cos t - C_8\sin t \\ u_1'''(t) &= 6\sqrt{6}C_5\sin\sqrt{6}t - 6\sqrt{6}C_6\cos\sqrt{6}t + C_7\sin t - C_8\cos t \end{aligned}$$

Apply the initial conditions now to determine C_5, C_6, C_7, and C_8.

$$\begin{aligned} u_1(0) &= C_5 + C_7 = 1 \\ u_1'(0) &= C_6\sqrt{6} + C_8 = 0 \\ u_1''(0) &= -6C_5 - C_7 = -1 \\ u_1'''(0) &= -6\sqrt{6}C_6 - C_8 = 0 \end{aligned}$$

Solving this system of equations yields $C_5 = 0$, $C_6 = 0$, $C_7 = 1$, and $C_8 = 0$. Therefore,

$$\boxed{u_1(t) = \cos t.}$$

Now that $u_1(t)$ is known, plug it back into equation (2) to find $u_2(t)$.

$$u_2'' + 2u_2 = 2\cos t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$u_2(t) = u_{2c}(t) + u_{2p}(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_{2c}'' + 2u_{2c} = 0 \tag{4}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u_{2c} = e^{st}$.

$$u_{2c} = e^{st} \quad \rightarrow \quad u_{2c}' = se^{st} \quad \rightarrow \quad u_{2c}'' = s^2e^{st}$$

Substitute these expressions into the ODE.

$$s^2e^{st} + 2(e^{st}) = 0$$

Divide both sides by e^{st}.

$$s^2 + 2 = 0$$

$$s = \{-i\sqrt{2}, i\sqrt{2}\}$$

As a result, two solutions to equation (4) are $u_{2c} = e^{-i\sqrt{2}t}$ and $u_{2c} = e^{i\sqrt{2}t}$. By the principle of superposition, the general solution for u_{2c} is a linear combination of these two.

$$\begin{aligned} u_{2c}(t) &= C_9 e^{-i\sqrt{2}t} + C_{10} e^{i\sqrt{2}t} \\ &= C_9(\cos\sqrt{2}t - i\sin\sqrt{2}t) + C_{10}(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ &= (C_9 + C_{10})\cos\sqrt{2}t + (-iC_9 + iC_{10})\sin\sqrt{2}t \\ &= C_{11}\cos\sqrt{2}t + C_{12}\sin\sqrt{2}t \end{aligned}$$

On the other hand, the particular solution satisfies

$$u_{2p}'' + 2u_{2p} = 2\cos t.$$

Since the inhomogeneous term is cosine and there are only even derivatives on the left, we use the trial solution $u_{2p} = A\cos t$. Plug this into the ODE to determine A.

$$(A\cos t)'' + 2(A\cos t) = 2\cos t$$

$$(-A\cos t) + 2A\cos t = 2\cos t$$

$$A\cos t = 2\cos t$$

Matching the coefficients, we see that $A = 2$, which means $u_{2p}(t) = 2\cos t$. The general solution for $u_2(t)$ is then

$$u_2(t) = C_{11}\cos\sqrt{2}t + C_{12}\sin\sqrt{2}t + 2\cos t.$$

Differentiate it with respect to t.

$$u_2'(t) = -C_{11}\sqrt{2}\sin\sqrt{2}t + C_{12}\sqrt{2}\cos\sqrt{2}t - 2\sin t$$

Apply the initial conditions to determine C_{11} and C_{12}.

$$u_2(0) = C_{11} + 2 = 2$$

$$u_2'(0) = C_{12}\sqrt{2} = 0$$

Solving this system of equations yields $C_{11} = 0$ and $C_{12} = 0$. Therefore,

$$\boxed{u_2(t) = 2\cos t.}$$

Part (d)

Suppose secondly that the initial conditions are

$$\begin{aligned} u_1(0) &= -2 & \qquad u_2(0) &= 1 \\ u_1'(0) &= 0 & \qquad u_2'(0) &= 0. \end{aligned}$$

From equation (1), $u_1''(0)$ and $u_1'''(0)$ can be determined.

$$u_1'' + 5u_1 = 2u_2 \quad \rightarrow \quad u_1'' = 2u_2 - 5u_1 \quad \rightarrow \quad \begin{cases} u_1''(0) = 2u_2(0) - 5u_1(0) = 12 \\ u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 0 \end{cases}$$

Differentiate the general solution three times.

$$\begin{aligned} u_1'(t) &= -C_5\sqrt{6}\sin\sqrt{6}t + C_6\sqrt{6}\cos\sqrt{6}t - C_7\sin t + C_8\cos t \\ u_1''(t) &= -6C_5\cos\sqrt{6}t - 6C_6\sin\sqrt{6}t - C_7\cos t - C_8\sin t \\ u_1'''(t) &= 6\sqrt{6}C_5\sin\sqrt{6}t - 6\sqrt{6}C_6\cos\sqrt{6}t + C_7\sin t - C_8\cos t \end{aligned}$$

Apply the initial conditions now to determine C_5, C_6, C_7, and C_8.

$$\begin{aligned} u_1(0) &= C_5 + C_7 = -2 \\ u_1'(0) &= C_6\sqrt{6} + C_8 = 0 \\ u_1''(0) &= -6C_5 - C_7 = 12 \\ u_1'''(0) &= -6\sqrt{6}C_6 - C_8 = 0 \end{aligned}$$

Solving this system of equations yields $C_5 = -2$, $C_6 = 0$, $C_7 = 0$, and $C_8 = 0$. Therefore,

$$\boxed{u_1(t) = -2\cos\sqrt{6}t.}$$

Now that $u_1(t)$ is known, plug it back into equation (2) to find $u_2(t)$.

$$u_2'' + 2u_2 = -4\cos\sqrt{6}t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$u_2(t) = u_{2c}(t) + u_{2p}(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_{2c}'' + 2u_{2c} = 0 \tag{4}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u_{2c} = e^{st}$.

$$u_{2c} = e^{st} \quad \rightarrow \quad u_{2c}' = se^{st} \quad \rightarrow \quad u_{2c}'' = s^2e^{st}$$

Substitute these expressions into the ODE.

$$s^2e^{st} + 2(e^{st}) = 0$$

Divide both sides by e^{st}.

$$s^2 + 2 = 0$$

$$s = \{-i\sqrt{2}, i\sqrt{2}\}$$

As a result, two solutions to equation (4) are $u_{2c} = e^{-i\sqrt{2}t}$ and $u_{2c} = e^{i\sqrt{2}t}$. By the principle of superposition, the general solution for u_{2c} is a linear combination of these two.

$$\begin{aligned} u_{2c}(t) &= C_9 e^{-i\sqrt{2}t} + C_{10} e^{i\sqrt{2}t} \\ &= C_9(\cos\sqrt{2}t - i\sin\sqrt{2}t) + C_{10}(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ &= (C_9 + C_{10})\cos\sqrt{2}t + (-iC_9 + iC_{10})\sin\sqrt{2}t \\ &= C_{11}\cos\sqrt{2}t + C_{12}\sin\sqrt{2}t \end{aligned}$$

On the other hand, the particular solution satisfies

$$u_{2p}'' + 2u_{2p} = -4\cos\sqrt{6}t.$$

Since the inhomogeneous term is cosine and there are only even derivatives on the left, we use the trial solution $u_{2p} = A\cos\sqrt{6}t$. Plug this into the ODE to determine A.

$$(A\cos\sqrt{6}t)'' + 2(A\cos\sqrt{6}t) = -4\cos\sqrt{6}t$$

$$(-6A\cos\sqrt{6}t) + 2A\cos\sqrt{6}t = -4\cos\sqrt{6}t$$

$$-4A\cos\sqrt{6}t = -4\cos\sqrt{6}t$$

Matching the coefficients, we see that $A = 1$, which means $u_{2p}(t) = \cos\sqrt{6}t$. The general solution for $u_2(t)$ is then

$$u_2(t) = C_{11}\cos\sqrt{2}t + C_{12}\sin\sqrt{2}t + \cos\sqrt{6}t.$$

Differentiate it with respect to t.

$$u_2'(t) = -C_{11}\sqrt{2}\sin\sqrt{2}t + C_{12}\sqrt{2}\cos\sqrt{2}t - \sqrt{6}\sin\sqrt{6}t$$

Apply the initial conditions to determine C_{11} and C_{12}.

$$\begin{aligned} u_2(0) &= C_{11} + 1 = 1 \\ u_2'(0) &= C_{12}\sqrt{2} = 0 \end{aligned}$$

Solving this system of equations yields $C_{11} = 0$ and $C_{12} = 0$. Therefore,

$$\boxed{u_2(t) = \cos\sqrt{6}t.}$$