Problem 11

In each of Problems 11 through 28, find the general solution of the given differential equation.

\[ y''' - y'' - y' + y = 0 \]

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2 e^{rt} \rightarrow y''' = r^3 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^3 e^{rt} - (r^2 e^{rt}) - (re^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^3 - r^2 - r + 1 = 0 \]

\[ (r + 1)(r - 1)^2 = 0 \]

\[ r = \{ -1, 1 \} \]

Two solutions to the ODE are then \( y = e^{-t} \) and \( y = e^t \). By using the method of reduction of order, we can obtain the general solution. Plug in \( y(t) = c(t)e^t \) to the ODE.

\[ [c(t)e^t]''' - [c(t)e^t]'' - [c(t)e^t]' + [c(t)e^t] = 0 \]

Evaluate the derivatives.

\[ [c'(t)e^t + c(t)e^t]' - [c'(t)e^t + c(t)e^t]' - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0 \]

\[ [c''(t)e^t + 2c'(t)e^t + c(t)e^t]' - [c''(t)e^t + 2c'(t)e^t + c(t)e^t] - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0 \]

\[ [c'''(t)e^t + 3c''(t)e^t + 3c'(t)e^t + c(t)e^t] - [c''(t)e^t + 2c'(t)e^t + c(t)e^t] - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0 \]

Expand the left side.

\[ c'''(t)e^t + 3c''(t)e^t + 3c'(t)e^t + c(t)e^t - c''(t)e^t - 2c'(t)e^t - c(t)e^t - c'(t)e^t - c(t)e^t + c(t)e^t = 0 \]

\[ c'''(t)e^t + 2c''(t)e^t = 0 \]

Bring the second term to the left side and then divide both sides by \( c''(t)e^t \).

\[ \frac{c'''(t)}{c''(t)} = -2 \]

The left side can be written as the derivative of a logarithm by the chain rule.

\[ \frac{d}{dt} \ln |c''(t)| = -2 \]

An absolute value sign has been included because the logarithm argument cannot be negative. Integrate both sides with respect to \( t \).

\[ \ln |c''(t)| = -2t + C_1 \]
Exponentiate both sides.

\[ |c''(t)| = e^{-2t+C_1} \]
\[ = e^{C_1}e^{-2t} \]

Place ± on the right side to remove the absolute value sign on the left.

\[ c''(t) = \pm e^{C_1}e^{-2t} \]

Use a new constant \( C_2 \) for \( \pm e^{C_1} \).

\[ c''(t) = C_2e^{-2t} \]

Integrate both sides with respect to \( t \) again.

\[ c'(t) = -\frac{C_2}{2}e^{-2t} + C_3 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = \frac{C_2}{4}e^{-2t} + C_3 t + C_4 \]

Therefore, since \( y(t) = c(t)e^t \),

\[ y(t) = C_5e^{-t} + C_3te^t + C_4e^t, \]

where a new constant \( C_5 \) is used for \( C_2/4 \).