Problem 14

In each of Problems 11 through 28, find the general solution of the given differential equation.

\[ y^{(4)} - 4y''' + 4y'' = 0 \]

Solution

Start by making the substitution \( u = y'' \). Then this fourth-order ODE reduces to a second-order ODE.

\[ u'' - 4u' + 4u = 0 \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( u = e^{rt} \).

\[ u = e^{rt} \rightarrow u' = re^{rt} \rightarrow u'' = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} - 4(re^{rt}) + 4(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 - 4r + 4 = 0 \]

\[ (r - 2)^2 = 0 \]

\[ r = \{2\} \]

One solution to the ODE is then \( u = e^{2t} \). By using the method of reduction of order, we can obtain the general solution. Plug in \( u(t) = c(t)e^{2t} \) to the ODE.

\[ [c(t)e^{2t}]'' - 4[c(t)e^{2t}]' + 4[c(t)e^{2t}] = 0 \]

Evaluate the derivatives.

\[ [c'(t)e^{2t} + 2c(t)e^{2t}]' - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4[c(t)e^{2t}] = 0 \]

\[ [c''(t)e^{2t} + 4c'(t)e^{2t} + 4c(t)e^{2t}] - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4[c(t)e^{2t}] = 0 \]

Expand the left side.

\[ c''(t)e^{2t} + 4c'(t)e^{2t} + 4c(t)e^{2t} - 4c'(t)e^{2t} - 8c(t)e^{2t} + 4c(t)e^{2t} = 0 \]

\[ c''(t)e^{t} = 0 \]

Divide both sides by \( e^t \).

\[ c''(t) = 0 \]

Integrate both sides with respect to \( t \).

\[ c'(t) = C_1 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = C_1t + C_2 \]

Since \( u(t) = c(t)e^{2t} \),

\[ u(t) = C_1te^{2t} + C_2e^{2t} \]

www.stemjock.com
Replace $u(t)$ with $y''$ now.

$$y''(t) = C_1 t e^{2t} + C_2 e^{2t}$$

Integrate both sides with respect to $t$.

$$y'(t) = \frac{C_1}{4} (2t - 1)e^{2t} + \frac{C_2}{2} e^{2t} + C_3$$

Integrate both sides with respect to $t$ once more.

$$y(t) = \frac{C_1}{4} (t - 1)e^{2t} + \frac{C_2}{4} e^{2t} + C_3 t + C_4$$

$$= \frac{C_1}{4} e^{2t} - \frac{C_1}{4} e^{2t} + \frac{C_2}{4} e^{2t} + C_3 t + C_4$$

$$= \frac{C_1}{4} e^{2t} + \frac{1}{4} (-C_1 + C_2) e^{2t} + C_3 t + C_4$$

Therefore, using $C_5$ for $C_1/4$ and $C_6$ for $(-C_1 + C_2)/4$, the general solution is

$$y(t) = C_5 t e^{2t} + C_6 e^{2t} + C_3 t + C_4.$$