Problem 19

In each of Problems 11 through 28, find the general solution of the given differential equation.

\[ y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y = 0 \]

Solution

Start by making the substitution \( u = y' \).

\[ u^{(4)} - 3u''' + 3u'' - 3u' + 2u = 0 \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( u = e^{rt} \).

\[ u = e^{rt} \rightarrow u' = re^{rt} \rightarrow u'' = r^2 e^{rt} \rightarrow u''' = r^3 e^{rt} \rightarrow u^{(4)} = r^4 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^4 e^{rt} - 3(r^3 e^{rt}) + 3(r^2 e^{rt}) - 3(re^{rt}) + 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^4 - 3r^3 + 3r^2 - 3r + 2 = 0 \]

\[ (r - 1)(r - 2)(r^2 + 1) = 0 \]

\[ r = \{1, 2, -i, i\} \]

Four solutions to the ODE are then \( y = e^t \) and \( y = e^{2t} \) and \( y = e^{it} \) and \( y = e^{-it} \). By the principle of superposition, the general solution for \( u \) is a linear combination of these four.

\[ u(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{it} + C_4 e^{-it} \]

Change back to \( y \) now.

\[ y'(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{it} + C_4 e^{-it} \]

Integrate both sides with respect to \( t \).

\[ y(t) = C_1 e^t + \frac{C_2}{2} e^{2t} + \frac{C_3}{i} e^{it} - \frac{C_4}{i} e^{-it} + C_5 \]

\[ = C_1 e^t + \frac{C_2}{2} e^{2t} + \frac{C_3}{i} (\cos t + i \sin t) - \frac{C_4}{i} (\cos t - i \sin t) + C_5 \]

\[ = C_1 e^t + \frac{C_2}{2} e^{2t} + \left( \frac{C_3}{i} - \frac{C_4}{i} \right) \cos t + (C_3 + C_4) \sin t + C_5 \]

Therefore, using new constants,

\[ y(t) = C_1 e^t + C_6 e^{2t} + C_7 \cos t + C_8 \sin t + C_5. \]