Problem 22

In each of Problems 11 through 28, find the general solution of the given differential equation.

\[ y^{(4)} + 2y'' + y = 0 \]

**Solution**

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt} \quad \rightarrow \quad y^{(4)} = r^4e^{rt} \]

Substitute these expressions into the ODE.

\[ r^4e^{rt} + 2(r^2e^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^4 + 2r^2 + 1 = 0 \]

\[ (r^2 + 1)^2 = 0 \] \quad (1)

\[ r^2 + 1 = 0 \]

\[ r = \{-i, i\} \]

Two solutions to the ODE are then \( y = e^{-it} \) and \( y = e^{it} \). The multiplicity of each of these roots is 2 because the quantity in equation (1) is squared. That means a second linearly independent solution can be obtained from each of these two by including a factor of \( t \): \( y = te^{-it} \) and \( y = te^{it} \).

By the principle of superposition, the general solution for \( y \) is a linear combination of these four.

\[
y(t) = C_1e^{-it} + C_2e^{it} + C_3te^{-it} + C_4te^{it}
\]

\[
= C_1(\cos t - i \sin t) + C_2(\cos t + i \sin t) + C_3t(\cos t - i \sin t) + C_4t(\cos t + i \sin t)
\]

\[
= (C_1 + C_2) \cos t + (C_3 + C_4)t \cos t + (-iC_1 + iC_2) \sin t + (-iC_3 + iC_4)t \sin t
\]

Therefore, using new arbitrary constants,

\[
y(t) = C_5 \cos t + C_6 t \cos t + C_7 \sin t + C_8 t \sin t.
\]