Problem 32

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y''' - y'' + y' - y = 0; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2 e^{rt} \rightarrow y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - r^2 e^{rt} + re^{rt} - e^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^3 - r^2 + r - 1 = 0$$

$$(r - 1)(r^2 + 1) = 0$$

$$r = \{1, -i, i\}$$

Three solutions to the ODE are then $y = e^{t}$ and $y = e^{-it}$ and $y = e^{it}$. By the principle of superposition, the general solution for $y$ is a linear combination of these three.

$$y(t) = C_1 e^{t} + C_2 e^{-it} + C_3 e^{it}$$

$$= C_1 e^{t} + C_2 (\cos t - i \sin t) + C_3 (\cos t + i \sin t)$$

$$= C_1 e^{t} + (C_2 + C_3) \cos t + (-iC_2 + iC_3) \sin t$$

$$= C_1 e^{t} + C_4 \cos t + C_5 \sin t$$

Differentiate this solution twice with respect to $t$.

$$y'(t) = C_1 e^{t} - C_4 \sin t + C_5 \cos t$$

$$y''(t) = C_1 e^{t} - C_4 \cos t - C_5 \sin t$$

Apply the initial conditions now to determine $C_1$, $C_4$, and $C_5$.

$$y(0) = C_1 + C_4 = 2$$

$$y'(0) = C_1 + C_5 = -1$$

$$y''(0) = C_1 - C_4 = -2$$

Solving this system of equations yields $C_1 = 0$, $C_4 = 2$, and $C_5 = -1$. Therefore,

$$y(t) = 2 \cos t - \sin t.$$
The solution oscillates like this for all $t$. 