Problem 33

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as \( t \to \infty \)?

\[ 2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0 \]

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y = e^{rt} \).

\[
\begin{align*}
y &= e^{rt} \quad \rightarrow \quad y' &= re^{rt} \quad \rightarrow \quad y'' &= r^2 e^{rt} \quad \rightarrow \quad y''' &= r^3 e^{rt} \quad \rightarrow \quad y^{(4)} &= r^4 e^{rt}
\end{align*}
\]

Substitute these expressions into the ODE.

\[
2(r^4 e^{rt}) - r^3 e^{rt} - 9(r^2 e^{rt}) + 4(re^{rt}) + 4(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
2r^4 - r^3 - 9r^2 + 4r + 4 = 0
\]

\[
(r + 2)(2r + 1)(r - 1)(r - 2) = 0
\]

\[
r = \left\{ -2, -\frac{1}{2}, 1, 2 \right\}
\]

Four solutions to the ODE are then \( y = e^{-2t} \) and \( y = e^{-t/2} \) and \( y = e^t \) and \( y = e^{2t} \). By the principle of superposition, the general solution for \( y \) is a linear combination of these four.

\[
y(t) = C_1 e^{-2t} + C_2 e^{-t/2} + C_3 e^t + C_4 e^{2t}
\]

Differentiate this solution three times with respect to \( t \).

\[
\begin{align*}
y'(t) &= -2C_1 e^{-2t} - \frac{C_2}{2} e^{-t/2} + C_3 e^t + 2C_4 e^{2t} \\
y''(t) &= 4C_1 e^{-2t} + \frac{C_2}{4} e^{-t/2} + C_3 e^t + 4C_4 e^{2t} \\
y'''(t) &= -8C_1 e^{-2t} - \frac{C_2}{8} e^{-t/2} + C_3 e^t + 8C_4 e^{2t}
\end{align*}
\]

Apply the initial conditions now to determine \( C_1, C_2, C_3, \) and \( C_4 \).

\[
\begin{align*}
y(0) &= C_1 + C_2 + C_3 + C_4 = -2 \\
y'(0) &= -2C_1 - \frac{C_2}{2} + C_3 + 2C_4 = 0 \\
y''(0) &= 4C_1 + \frac{C_2}{4} + C_3 + 4C_4 = -2 \\
y'''(0) &= -8C_1 - \frac{C_2}{8} + C_3 + 8C_4 = 0
\end{align*}
\]

Solving this system of equations yields \( C_1 = -1/6, \ C_2 = -16/15, \ C_3 = -2/3, \) and \( C_4 = -1/10 \). Therefore,

\[
y(t) = -\frac{1}{6} e^{-2t} - \frac{16}{15} e^{-t/2} - \frac{2}{3} e^t - \frac{1}{10} e^{2t}.
\]
The first two terms tend to zero in the limit as $t \to \infty$, but the other two terms blow up and make $y(t) \to -\infty$ in the limit.