Problem 35

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as \( t \to \infty \)?

\[
6y''' + 5y'' + y' = 0; \quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0
\]

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y = e^{rt} \).

\[
y = e^{rt} \quad \Rightarrow \quad y' = re^{rt} \quad \Rightarrow \quad y'' = r^2 e^{rt} \quad \Rightarrow \quad y''' = r^3 e^{rt}
\]

Substitute these expressions into the ODE.

\[
6(r^3 e^{rt}) + 5(r^2 e^{rt}) + re^{rt} = 0
\]

Divide both sides by \( e^{rt} \).

\[
6r^3 + 5r^2 + r = 0 \quad \Rightarrow \quad r(2r + 1)(3r + 1) = 0 \quad \Rightarrow \quad r = \left\{ -\frac{1}{2}, -\frac{1}{3}, 0 \right\}
\]

Three solutions to the ODE are then \( y = e^{-t/2} \) and \( y = e^{-t/3} \) and \( e^0 = 1 \). By the principle of superposition, the general solution is a linear combination of these three.

\[
y(t) = C_1 e^{-t/2} + C_2 e^{-t/3} + C_3
\]

Differentiate it twice with respect to \( t \).

\[
y'(t) = -\frac{C_1}{2} e^{-t/2} - \frac{C_2}{3} e^{-t/3} \\
y''(t) = \frac{C_1}{4} e^{-t/2} + \frac{C_2}{9} e^{-t/3}
\]

Apply the initial conditions now to determine \( C_1, C_2, \) and \( C_3 \).

\[
y(0) = C_1 + C_2 + C_3 = -2 \\
y'(0) = -\frac{C_1}{2} - \frac{C_2}{3} = 2 \\
y''(0) = \frac{C_1}{4} + \frac{C_2}{9} = 0
\]

Solving this system of equations yields \( C_1 = 8, \ C_2 = -18, \) and \( C_3 = 8 \). Therefore,

\[
y(t) = 8e^{-t/2} - 18e^{-t/3} + 8.
\]
\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} (8e^{-t/2} - 18e^{-t/3} + 8) = 8
\]