Problem 37

Show that the general solution of \( y^{(4)} - y = 0 \) can be written as

\[
y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.
\]

Determine the solution satisfying the initial conditions \( y(0) = 0, y'(0) = 0, y''(0) = 1, y'''(0) = 1 \). Why is it convenient to use the solutions \( \cosh t \) and \( \sinh t \) rather than \( e^t \) and \( e^{-t} \)?

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y = e^{rt} \).

\[
y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2 e^{rt} \rightarrow y''' = r^3 e^{rt} \rightarrow y^{(4)} = r^4 e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^4 e^{rt} - e^{rt} = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^4 - 1 = 0
\]

\[
(r + 1)(r - 1)(r^2 + 1) = 0
\]

\[
r = \{ -1, 1, -i, i \}
\]

Four solutions to the ODE are then \( y = e^{-t} \) and \( y = e^t \) and \( y = e^{-it} \) and \( y = e^{it} \). By the principle of superposition, the general solution for \( y \) is a linear combination of these four.

\[
y(t) = C_1 e^{-t} + C_2 e^t + C_3 e^{-it} + C_4 e^{it}
\]

Note that hyperbolic sine and hyperbolic cosine are defined as

\[
cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}.
\]

Adding the respective sides of these equations yields

\[
cosh t + \sinh t = e^t.
\]

On the other hand, subtracting the respective sides of these equations yields

\[
cosh t - \sinh t = e^{-t}.
\]

Substitute these two previous equations and use Euler's formula in the general solution.

\[
y(t) = C_1 (\cosh t - \sinh t) + C_2 (\cosh t + \sinh t) + C_3 (\cos t - i \sin t) + C_4 (\cos t + i \sin t)
\]

\[
= (C_1 + C_2) \cosh t + (-C_1 + C_2) \sinh t + (C_3 + C_4) \cos t + (-iC_3 + iC_4) \sin t
\]

Therefore, using new arbitrary constants,

\[
y(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.
\]

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Differentiate this solution three times with respect to $t$.

\[
\begin{align*}
    y'(t) &= -c_1 \sin t + c_2 \cos t + c_3 \sinh t + c_4 \cosh t \\
    y''(t) &= -c_1 \cos t - c_2 \sin t + c_3 \cosh t + c_4 \sinh t \\
    y'''(t) &= c_1 \sin t - c_2 \cos t + c_3 \sinh t + c_4 \cosh t
\end{align*}
\]

Apply the initial conditions now to determine $c_1$, $c_2$, $c_3$, and $c_4$.

\[
\begin{align*}
    y(0) &= c_1 + c_3 = 0 \\
    y'(0) &= c_2 + c_4 = 0 \\
    y''(0) &= -c_1 + c_3 = 1 \\
    y'''(0) &= -c_2 + c_4 = 1
\end{align*}
\]

Solving this system of equations yields $c_1 = -1/2$, $c_2 = -1/2$, $c_3 = 1/2$, and $c_4 = 1/2$. Therefore,

\[
\begin{align*}
    y(t) &= -\frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{1}{2} \cosh t + \frac{1}{2} \sinh t \\
    &= \frac{1}{2} (\cosh t + \sinh t - \cos t - \sin t).
\end{align*}
\]