Problem 1

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y''' - y'' - y' + y = 2e^{-t} + 3$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c'' - y_c' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - r^2e^{rt} - re^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r^2 - r + 1 = 0$$

$$(r+1)(r-1)^2 = 0$$

$$r = \{-1, 1\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^t$. Since the multiplicity of the $r = 1$ root is 2, a second linearly independent solution can be obtained from it by including a factor of t: $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2e^t + C_3te^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p'' - y_p' + y_p = 2e^{-t} + 3.$$

The right side has two terms. To account for the first one, we would include Ae^{-t} in the trial solution, but since e^{-t} already satisfies y_c, we will include Ate^{-t}. To account for the second one, we will include B in the trial solution. Substitute $y_p(t) = Ate^{-t} + B$ in the ODE to determine A and B.

$$(Ate^{-t} + B)''' - (Ate^{-t} + B)'' - (Ate^{-t} + B)' + (Ate^{-t} + B) = 2e^{-t} + 3$$

Evaluate the derivatives.

$$(Ae^{-t} - Ate^{-t})'' - (Ae^{-t} - Ate^{-t})' - (Ae^{-t} - Ate^{-t}) + (Ate^{-t} + B) = 2e^{-t} + 3$$

$$(-Ae^{-t} - Ae^{-t} + Ate^{-t})' - (-Ae^{-t} - Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) + (Ate^{-t} + B) = 2e^{-t} + 3$$

$$(Ae^{-t} + Ae^{-t} + Ae^{-t} - Ate^{-t}) - (-Ae^{-t} - Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) + (Ate^{-t} + B) = 2e^{-t} + 3$$

Simplify the left side.

$$4Ae^{-t} + B = 2e^{-t} + 3$$

Match the coefficients to obtain a system of equations for A and B.

$$\begin{aligned} 4A &= 2 \\ B &= 3 \end{aligned}$$

Solving this system yields $A = 1/2$ and $B = 3$. As a result, the particular solution is $y_p(t) = (1/2)te^{-t} + 3$, and the general solution is

$$y(t) = C_1e^{-t} + C_2e^t + C_3te^t + \frac{1}{2}te^{-t} + 3.$$