Problem 2

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y^{(4)} - y = 3t + \cos t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - 1 = 0$$
$$(r^2+1)(r^2-1) = 0$$
$$r = \{-i, i, -1, 1\}$$

Four solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$ and $y_c = e^{-t}$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1e^{-it} + C_2e^{it} + C_3e^{-t} + C_4e^t \\ &= C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t) + C_3e^{-t} + C_4e^t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + C_3e^{-t} + C_4e^t \\ &= C_5\cos t + C_6\sin t + C_3e^{-t} + C_4e^t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - y_p = 3t + \cos t.$$

The right side has two terms. To account for the first one, we will include $A + Bt$ in the trial solution. To account for the second one, we would include $C\cos t + D\sin t$ in the trial solution, but because $\cos t$ satisfies equation (1), an extra factor of t is needed. Substitute $y_p(t) = A + Bt + Ct\cos t + Dt\sin t$ in the ODE to determine A, B, C, and D.

$$(A + Bt + Ct\cos t + Dt\sin t)^{(4)} - (A + Bt + Ct\cos t + Dt\sin t) = 3t + \cos t$$

Evaluate the derivatives.

$$[(Ct - 4D)\cos t + (4C + Dt)\sin t] - (A + Bt + Ct\cos t + Dt\sin t) = 3t + \cos t$$

Simplify the left side.

$$-A - Bt + 4C\sin t - 4D\cos t = 3t + \cos t$$

Match the coefficients to obtain a system of equations for A and B.

$$\begin{aligned} -A &= 0 \\ -B &= 3 \\ 4C &= 0 \\ -4D &= 1 \end{aligned}$$

Solving this system yields $A = 0$, $B = -3$, $C = 0$, and $D = -1/4$. As a result, the particular solution is $y_p(t) = -3t - (1/4)t\sin t$, and the general solution is

$$y(t) = C_5\cos t + C_6\sin t + C_3e^{-t} + C_4e^{t} - 3t - \frac{t\sin t}{4}.$$