Problem 3

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y''' + y'' + y' + y = e^{-t} + 4t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' + y_c'' + y_c' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} + r^2 e^{rt} + re^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$\begin{gathered} r^3 + r^2 + r + 1 = 0 \\ (r+1)(r^2+1) = 0 \\ r = \{-1, -i, i\} \end{gathered}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1 e^{-t} + C_2 e^{-it} + C_3 e^{it} \\ &= C_1 e^{-t} + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \\ &= C_1 e^{-t} + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\ &= C_1 e^{-t} + C_4 \cos t + C_5 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' + y_p'' + y_p' + y_p = e^{-t} + 4t.$$

The right side has two terms. To account for the first one, we would include Ae^{-t} in the trial solution, but since e^{-t} already satisfies y_c, we will include Ate^{-t}. To account for the second one, we will include $B + Ct$ in the trial solution. Substitute $y_p(t) = Ate^{-t} + B + Ct$ in the ODE to determine A and B and C.

$$(Ate^{-t} + B + Ct)''' + (Ate^{-t} + B + Ct)'' + (Ate^{-t} + B + Ct)' + (Ate^{-t} + B + Ct) = e^{-t} + 4t$$

Evaluate the derivatives.

$$(Ae^{-t} - Ate^{-t} + C)'' + (Ae^{-t} - Ate^{-t} + C)' + (Ae^{-t} - Ate^{-t} + C) + (Ate^{-t} + B + Ct) = e^{-t} + 4t$$

$(-Ae^{-t}-Ae^{-t}+Ate^{-t})'+(-Ae^{-t}-Ae^{-t}+Ate^{-t})+(Ae^{-t}-Ate^{-t}+C)+(Ate^{-t}+B+Ct)=e^{-t}+4t$

$(Ae^{-t}+Ae^{-t}+Ae^{-t}-Ate^{-t})+(-Ae^{-t}-Ae^{-t}+Ate^{-t})+(Ae^{-t}-Ate^{-t}+C)+(Ate^{-t}+B+Ct)=e^{-t}+4t$

Simplify the left side.

$$2Ae^{-t}+B+C+Ct=e^{-t}+4t$$

Match the coefficients to obtain a system of equations for A, B, and C.

$$\begin{aligned} 2A &= 1 \\ B+C &= 0 \\ C &= 4 \end{aligned}$$

Solving this system yields $A=1/2$, $B=-4$, and $C=4$. As a result, the particular solution is $y_p(t)=(1/2)te^{-t}-4+4t$, and the general solution is

$$y(t)=C_1e^{-t}+C_4\cos t+C_5\sin t+\frac{1}{2}te^{-t}-4+4t.$$