Problem 8

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y^{(4)} + y''' = \sin 2t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + y_c''' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + r^3e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 + r^3 = 0$$

$$r^3(r+1) = 0$$

$$r = \{-1, 0\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$. Since the multiplicity of the $r = 0$ root is 3, a second and third linearly independent solution can be obtained from the first by including factors of t and t^2: $y_c = te^0 = t$ and $y_c = t^2e^0 = t^2$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$y_c(t) = C_1e^{-t} + C_2 + C_3t + C_4t^2$$

On the other hand, the particular solution satisfies

$$y_p^{(6)} + y_p''' = \sin 2t.$$

To account for the inhomogeneous term, we will include $A\sin 2t + B\cos 2t$ in the trial solution. Substitute $y_p(t) = A\sin 2t + B\cos 2t$ in the ODE to determine A and B.

$$(A\sin 2t + B\cos 2t)^{(4)} + (A\sin 2t + B\cos 2t)''' = \sin 2t$$

Evaluate the derivatives.

$$(16A\sin 2t + 16B\cos 2t) + (-8A\cos 2t + 8B\sin 2t) = \sin 2t$$

Simplify the left side.

$$(16A + 8B)\sin 2t + (16B - 8A)\cos 2t = \sin 2t$$

Match the coefficients to obtain a system of equations for A and B.

$$\begin{aligned} 16A + 8B &= 1 \\ 16B - 8A &= 0 \end{aligned}$$

Solving this system yields $A = 1/20$ and $B = 1/40$. As a result, the particular solution is $y_p(t) = (1/20)\sin 2t + (1/40)\cos 2t$, and the general solution is

$$y(t) = C_1 e^{-t} + C_2 + C_3 t + C_4 t^2 + \frac{1}{20}\sin 2t + \frac{1}{40}\cos 2t.$$