Problem 12

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

$$y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}; \qquad y(0) = 3, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 2$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c''' + y_c'' + 8y_c' - 12y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 2(r^3e^{rt}) + r^2e^{rt} + 8(re^{rt}) - 12(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 2r^3 + r^2 + 8r - 12 = 0$$

$$(r + 3)(r - 1)(r^2 + 4) = 0$$

$$r = \{-3, 1, -2i, 2i\}$$

Four solutions to equation (1) are then $y_c = e^{-3t}$ and $y_c = e^t$ and $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned}
y_c(t) &= C_1e^{-3t} + C_2e^t + C_3e^{-2it} + C_4e^{2it} \\
&= C_1e^{-3t} + C_2e^t + C_3(\cos 2t - i\sin 2t) + C_4(\cos 2t + i\sin 2t) \\
&= C_1e^{-3t} + C_2e^t + (C_3 + C_4)\cos 2t + (-iC_3 + iC_4)\sin 2t \\
&= C_1e^{-3t} + C_2e^t + C_5\cos 2t + C_6\sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p''' + y_p'' + 8y_p' - 12y_p = 12\sin t - e^{-t}.$$

There are two terms on the right side. To account for the first one, we will include $A\sin t + B\cos t$ in the trial solution. To account for the second one, we will include Ce^{-t} in the trial solution. Substitute $y_p(t) = A\sin t + B\cos t + Ce^{-t}$ in the ODE to determine A, B, and C.

$$\begin{aligned}
&(A\sin t + B\cos t + Ce^{-t})^{(4)} + 2(A\sin t + B\cos t + Ce^{-t})''' + (A\sin t + B\cos t + Ce^{-t})'' \\
&\qquad + 8(A\sin t + B\cos t + Ce^{-t})' - 12(A\sin t + B\cos t + Ce^{-t}) = 12\sin t - e^{-t}
\end{aligned}$$

Evaluate the derivatives.

$$(A\sin t + B\cos t + Ce^{-t}) + 2(-A\cos t + B\sin t - Ce^{-t}) + (-A\sin t - B\cos t + Ce^{-t}) \\ + 8(A\cos t - B\sin t - Ce^{-t}) - 12(A\sin t + B\cos t + Ce^{-t}) = 12\sin t - e^{-t}$$

Simplify the left side.

$$(6A - 12B)\cos t + (-12A - 6B)\sin t - 20Ce^{-t} = 12\sin t - e^{-t}$$

Match the coefficients to obtain a system of equations for A, B, and C.

$$\begin{aligned} 6A - 12B &= 0 \\ -12A - 6B &= 12 \\ -20C &= -1 \end{aligned}$$

Solving this system yields $A = -4/5$, $B = -2/5$, and $C = 1/20$. As a result, the particular solution is $y_p(t) = (-4/5)\sin t - (2/5)\cos t + (1/20)e^{-t}$, and the general solution is

$$y(t) = C_1e^{-3t} + C_2e^t + C_5\cos 2t + C_6\sin 2t - \frac{4}{5}\sin t - \frac{2}{5}\cos t + \frac{1}{20}e^{-t}.$$

Differentiate it with respect to t three times.

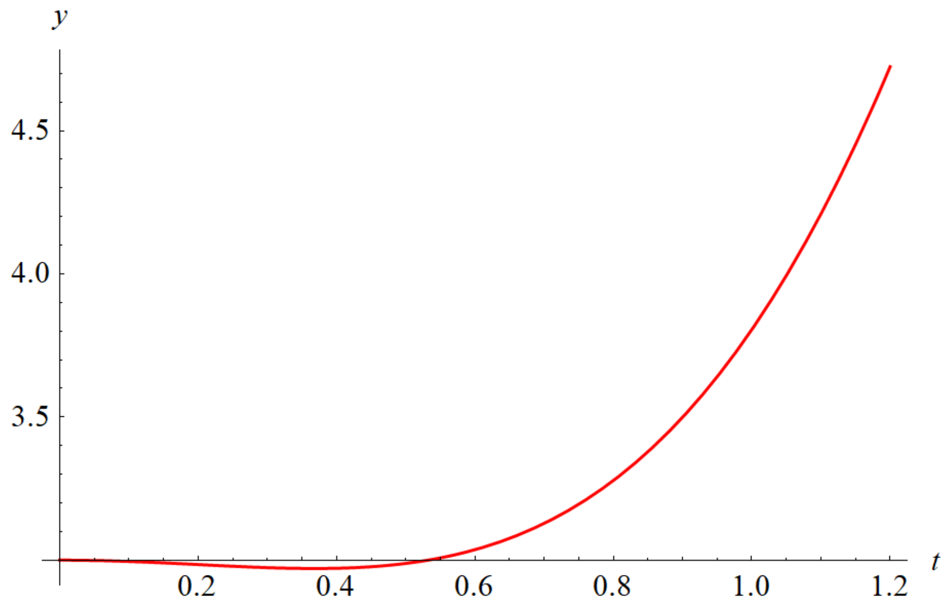
$$\begin{aligned} y'(t) &= -3C_1e^{-3t} + C_2e^t - 2C_5\sin 2t + 2C_6\cos 2t - \frac{4}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{20}e^{-t} \\ y''(t) &= 9C_1e^{-3t} + C_2e^t - 4C_5\cos 2t - 4C_6\sin 2t + \frac{4}{5}\sin t + \frac{2}{5}\cos t + \frac{1}{20}e^{-t} \\ y'''(t) &= -27C_1e^{-3t} + C_2e^t + 8C_5\sin 2t - 8C_6\cos 2t + \frac{4}{5}\cos t - \frac{2}{5}\sin t - \frac{1}{20}e^{-t} \end{aligned}$$

Now apply the initial conditions to determine C_1, C_2, C_5, and C_6.

$$\begin{aligned} y(0) &= C_1 + C_2 + C_5 - \frac{2}{5} + \frac{1}{20} = 3 \\ y'(0) &= -3C_1 + C_2 + 2C_6 - \frac{4}{5} - \frac{1}{20} = 0 \\ y''(0) &= 9C_1 + C_2 - 4C_5 + \frac{2}{5} + \frac{1}{20} = -1 \\ y'''(0) &= -27C_1 + C_2 - 8C_6 + \frac{4}{5} - \frac{1}{20} = 2 \end{aligned}$$

Solving this system of equations yields $C_1 = 73/520$, $C_2 = 81/40$, $C_5 = 77/65$, and $C_6 = -49/130$. Therefore,

$$y(t) = \frac{73}{520}e^{-3t} + \frac{81}{40}e^t + \frac{77}{65}\cos 2t - \frac{49}{130}\sin 2t - \frac{4}{5}\sin t - \frac{2}{5}\cos t + \frac{1}{20}e^{-t}.$$


y
4.5
4.0
3.5
0.2
0.4
0.6
0.8
1.0
1.2
t