Problem 14

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

$$y''' - y' = te^{-t} + 2\cos t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r = 0$$

$$r(r+1)(r-1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = te^{-t} + 2\cos t.$$

There are two terms on the right side. To account for the first one, we would include $(A + Bt)e^{-t}$ in the trial solution, but because e^{-t} satisfies equation (1), an extra factor of t is needed. To account for the second term, we will include $C\sin t$ since only odd derivatives are present on the left side. Therefore, the trial solution to plug in is

$$y_p(t) = t(A + Bt)e^{-t} + C\sin t.$$