Problem 16

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

$$y^{(4)} + 4y'' = \sin 2t + te^t + 4$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 4y_c'' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 4(r^2e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 4r^2 = 0$$

$$r^2(r^2 + 4) = 0$$

$$r = \{0, -2i, 2i\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{-2it}$ and $y_c = e^{2it}$. Since the multiplicity of the $r = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^0 = t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 + C_2t + C_3e^{-2it} + C_4e^{2it} \\ &= C_1 + C_2t + C_3(\cos 2t - i\sin 2t) + C_4(\cos 2t + i\sin 2t) \\ &= C_1 + C_2t + (C_3 + C_4)\cos 2t + (-iC_3 + iC_4)\sin 2t \\ &= C_1 + C_2t + C_5\cos 2t + C_6\sin 2t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 4y_p'' = \sin 2t + te^t + 4.$$

There are three terms on the right side. To account for the first one, we would include $A\sin 2t$ in the trial solution, but because $\sin 2t$ satisfies equation (1), cosine and an extra factor of t are needed. To account for the second term, we will include $(C + Dt)e^t$ in the trial solution. To account for the third term, we would include E in the trial solution, but because a constant and t satisfy equation (1), an extra factor of t^2 is needed. Therefore, the trial solution to plug in is

$$y_p(t) = t(A\sin 2t + B\cos 2t) + (C + Dt)e^t + Et^2.$$