Problem 17

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - y_c''' - y_c'' + y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - r^3e^{rt} - r^2e^{rt} + re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - r^3 - r^2 + r = 0$$

$$r(r+1)(r-1)^2 = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. Since the multiplicity of the $r = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t + C_4te^t$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - y_p''' - y_p'' + y_p' = t^2 + 4 + t\sin t.$$

There are three terms on the right side. To account for the first two, we would include $A + Bt + Ct^2$ in the trial solution, but because a constant satisfies equation (1), an extra factor of t is needed. To account for the third term, we will include $(D + Et)(F\cos t + G\sin t)$ in the trial solution. Therefore, the trial solution to plug in is

$$y_p(t) = t(A + Bt + Ct^2) + (D + Et)(F\cos t + G\sin t).$$