# Problem 21

Consider the problem of finding the form of a particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \tag{i}$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator $(D-2)(D+1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D-2)^4(D+1)^3Y = 0. \tag{ii}$$

Thus $Y$ is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t}, \tag{iii}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.

(c) Observe that $e^{2t}$, $te^{2t}$, $t^2e^{2t}$, and $e^{-t}$ are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose $c_1$, $c_2$, $c_3$, and $c_5$ to be zero in Eq. (iii), so that

$$Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}. \tag{iv}$$

This is the form of the particular solution $Y$ of Eq. (i). The values of the coefficients $c_4$, $c_6$, and $c_7$ can be found by substituting from Eq. (iv) in the differential equation (i).

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**Solution**

**Part (a)**

Show first that $D-2$ annihilates $3e^{2t}$.

$$\begin{aligned}(D-2)3e^{2t} &= D(3e^{2t}) - 2(3e^{2t})\\ &= 6e^{2t} - 6e^{2t}\\ &= 0\end{aligned}$$

Show secondly that $(D+1)^2$ annihilates $3e^{2t}$.

$$\begin{aligned}(D+1)^2(te^{-t}) &= (D^2+2D+1)(te^{-t})\\ &= D^2(te^{-t}) + 2D(te^{-t}) + 1(te^{-t})\\ &= D(e^{-t} - te^{-t}) + 2(e^{-t} - te^{-t}) + te^{-t}\\ &= (-e^{-t} - e^{-t} + te^{-t}) + 2(e^{-t} - te^{-t}) + te^{-t}\\ &= 0\end{aligned}$$

Show thirdly that $(D-2)(D+1)^2$ annihilates the right side of Eq. (i).

$$\begin{aligned}
(D-2)(D+1)^2(3e^{2t}-te^{-t}) &= (D^3-3D-2)(3e^{2t}-te^{-t}) \\
&= D^3(3e^{2t}-te^{-t}) - 3D(3e^{2t}-te^{-t}) - 2(3e^{2t}-te^{-t}) \\
&= D^2(6e^{2t}-e^{-t}+te^{-t}) - 3(6e^{2t}-e^{-t}+te^{-t}) - 2(3e^{2t}-te^{-t}) \\
&= D(12e^{2t}+e^{-t}+e^{-t}-te^{-t}) - 3(6e^{2t}-e^{-t}+te^{-t}) - 2(3e^{2t}-te^{-t}) \\
&= (24e^{2t}-e^{-t}-e^{-t}-e^{-t}+te^{-t}) - 3(6e^{2t}-e^{-t}+te^{-t}) - 2(3e^{2t}-te^{-t}) \\
&= 0
\end{aligned}$$

## Part (b)

$$(D-2)^3(D+1)Y = 3e^{2t}-te^{-t}$$

Apply the operator $(D-2)(D+1)^2$ to both sides.

$$(D-2)(D+1)^2(D-2)^3(D+1)Y = (D-2)(D+1)^2(3e^{2t}-te^{-t})$$

In Problem 20 it was shown that the operators are commutative. Also, the right side is zero by the last calculation in part (a).

$$(D-2)^4(D+1)^3Y = 0 \tag{ii}$$

$$(D^7-5D^6+3D^5+17D^4-16D^3-24D^2+16D+16)Y = 0$$

$$Y^{(7)}-5Y^{(6)}+3Y^{(5)}+17Y^{(4)}-16Y'''-24Y''+16Y'+16Y = 0$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form $Y = e^{rt}$.

$$Y = e^{rt} \quad \rightarrow \quad Y' = re^{rt} \quad \rightarrow \quad Y'' = r^2e^{rt} \quad \rightarrow \quad Y''' = r^3e^{rt} \quad \rightarrow \quad Y^{(7)} = r^7e^{rt}$$

Substitute these expressions into the ODE.

$$r^7e^{rt} - 5(r^6e^{rt}) + 3(r^5e^{rt}) + 17(r^4e^{rt}) - 16(r^3e^{rt}) - 24(r^2e^{rt}) + 16(re^{rt}) + 16(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$r^7-5r^6+3r^5+17r^4-16r^3-24r^2+16r+16 = 0$$

$$(r-2)^4(r+1)^3 = 0$$

$$r = \{-1, 2\}$$

Two solutions to Eq. (ii) are $Y = e^{-t}$ and $Y = e^{2t}$. The multiplicity of the $r = -1$ root is 3, so a second and third linearly independent solution can be obtained from the first by including factors of $t$ and $t^2$: $Y = te^{-t}$ and $Y = t^2e^{-t}$. The multiplicity of the $r = 2$ root is 4, so a second, third, and fourth linearly independent solution can be obtained from the first by including factors of $t$, $t^2$, and $t^3$: $Y = te^{2t}$ and $Y = t^2e^{2t}$ and $Y = t^3e^{2t}$. By the principle of superposition, the general solution for $Y$ is a linear combination of these seven.

$$Y(t) = C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + C_4e^{2t} + C_5te^{2t} + C_6t^2e^{2t} + C_7t^3e^{2t}$$

**Part (c)**

The associated homogeneous equation to

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t} \tag{i}$$

is

$$(D-2)^3(D+1)Y_c = 0$$
$$(D^4 - 5D^3 + 6D^2 + 4D - 8)Y_c = 0$$
$$Y_c^{(4)} - 5Y_c''' + 6Y_c'' + 4Y_c' - 8Y_c = 0.$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form $Y = e^{st}$.

$$Y_c = e^{st} \quad \to \quad Y_c' = se^{st} \quad \to \quad Y_c'' = s^2e^{st} \quad \to \quad Y_c''' = s^3e^{st} \quad \to \quad Y_c^{(4)} = s^4e^{st}$$

Substitute these expressions into the ODE.

$$s^4e^{st} - 5(s^3e^{st}) + 6(s^2e^{st}) + 4(se^{st}) - 8(e^{st}) = 0$$

Divide both sides by $e^{st}$.

$$s^4 - 5s^3 + 6s^2 + 4s - 8 = 0$$
$$(s-2)^3(s+1) = 0$$
$$s = \{-1, 2\}$$

Two solutions to the associated homogeneous equation are $Y = e^{-t}$ and $Y = e^{2t}$. The multiplicity of the $r = 2$ root is 3, so a second and third linearly independent solution can be obtained from the first by including factors of $t$ and $t^2$: $Y = te^{2t}$ and $Y = t^2e^{2t}$. By the principle of superposition, the general solution for $Y$ is a linear combination of these four.

$$Y_c(t) = C_8e^{-t} + C_9e^{2t} + C_{10}te^{2t} + C_{11}t^2e^{2t}$$

Comparing this to the result of part (b), we set $C_1 = 0$, $C_4 = 0$, $C_5 = 0$, and $C_6 = 0$ so that no terms are shared in common.

$$Y(t) = C_2te^{-t} + C_3t^2e^{-t} + C_7t^3e^{2t}$$