Problem 21

Consider the problem of finding the form of a particular solution \( Y(t) \) of

\[
(D - 2)^3(D + 1)Y = 3e^{2t} - te^{-t},
\]

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that \( D - 2 \) and \( (D + 1)^2 \), respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator \( (D - 2)(D + 1)^2 \) annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator \( (D - 2)(D + 1)^2 \) to Eq. (i) and use the result of Problem 20 to obtain

\[
(D - 2)^4(D + 1)^3Y = 0.
\]

Thus \( Y \) is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

\[
Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t},
\]

where \( c_1, \ldots, c_7 \) are constants, as yet undetermined.

(c) Observe that \( e^{2t}, te^{2t}, t^2e^{2t}, \) and \( e^{-t} \) are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose \( c_1, c_2, c_3, \) and \( c_5 \) to be zero in Eq. (iii), so that

\[
Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}.
\]

This is the form of the particular solution \( Y \) of Eq. (i). The values of the coefficients \( c_4, c_6, \) and \( c_7 \) can be found by substituting from Eq. (iv) in the differential equation (i).

Solution

Part (a)

Show first that \( D - 2 \) annihilates \( 3e^{2t} \).

\[
(D - 2)3e^{2t} = D(3e^{2t}) - 2(3e^{2t})
= 6e^{2t} - 6e^{2t}
= 0
\]

Show secondly that \( (D + 1)^2 \) annihilates \( 3e^{2t} \).

\[
(D + 1)^2(te^{-t}) = (D^2 + 2D + 1)(te^{-t})
= D^2(te^{-t}) + 2D(te^{-t}) + 1(te^{-t})
= D(e^{-t} - te^{-t}) + 2(e^{-t} - te^{-t}) + te^{-t}
= (-e^{-t} - e^{-t} + te^{-t}) + 2(e^{-t} - te^{-t}) + te^{-t}
= 0
\]
Show thirdly that \((D - 2)(D + 1)^2\) annihilates the right side of Eq. (i).
\[
(D - 2)(D + 1)^2(3e^{2t} - te^{-t}) = (D^3 - 3D - 2)(3e^{2t} - te^{-t})
\]
\[
= D^3(3e^{2t} - te^{-t}) - 3D(3e^{2t} - te^{-t}) - 2(3e^{2t} - te^{-t})
\]
\[
= D^2(6e^{2t} - e^{-t} + te^{-t}) - 3(6e^{2t} - e^{-t} + te^{-t}) - 2(3e^{2t} - te^{-t})
\]
\[
= D(12e^{2t} + e^{-t} + e^{-t} - te^{-t}) - 3(6e^{2t} - e^{-t} + te^{-t}) - 2(3e^{2t} - te^{-t})
\]
\[
= (24e^{2t} - e^{-t} - e^{-t} - e^{-t} + te^{-t}) - 3(6e^{2t} - e^{-t} + te^{-t}) - 2(3e^{2t} - te^{-t})
\]
\[
= 0
\]

Part (b)
\[
(D - 2)^3(D + 1)Y = 3e^{2t} - te^{-t}
\]
Apply the operator \((D - 2)(D + 1)^2\) to both sides.
\[
(D - 2)(D + 1)^2(D - 2)^3(D + 1)Y = (D - 2)(D + 1)^2(3e^{2t} - te^{-t})
\]
In Problem 20 it was shown that the operators are commutative. Also, the right side is zero by the last calculation in part (a).
\[
(D - 2)^4(D + 1)^3Y = 0
\]
\[
(D^7 - 5D^6 + 3D^5 + 17D^4 - 16D^3 - 24D^2 + 16D + 16)Y = 0
\]
\[
Y^{(7)} - 5Y^{(6)} + 3Y^{(5)} + 17Y^{(4)} - 16Y''' - 24Y'' + 16Y + 16 = 0
\]
This is a linear homogeneous ODE with constant coefficients, so the solution is of the form \(Y = e^{rt}\).
\[
Y = e^{rt} \rightarrow Y' = re^{rt} \rightarrow Y'' = r^2e^{rt} \rightarrow Y''' = r^3e^{rt} \rightarrow Y^{(7)} = r^7e^{rt}
\]
Substitute these expressions into the ODE.
\[
r^7e^{rt} - 5(r^6e^{rt}) + 3(r^5e^{rt}) + 17(r^4e^{rt}) - 16(r^3e^{rt}) - 24(r^2e^{rt}) + 16(re^{rt}) + 16(e^{rt}) = 0
\]
Divide both sides by \(e^{rt}\).
\[
r^7 - 5r^6 + 3r^5 + 17r^4 - 16r^3 - 24r^2 + 16r + 16 = 0
\]
\[
(r - 2)^4(r + 1)^3 = 0
\]
\[
r = \{-1, 2\}
\]
Two solutions to Eq. (ii) are \(Y = e^{-t}\) and \(Y = e^{2t}\). The multiplicity of the \(r = -1\) root is 3, so a second and third linearly independent solution can be obtained from the first by including factors of \(t\) and \(t^2\): \(Y = te^{-t}\) and \(Y = t^2e^{-t}\). The multiplicity of the \(r = 2\) root is 4, so a second, third, and fourth linearly independent solution can be obtained from the first by including factors of \(t\), \(t^2\), and \(t^3\): \(Y = te^{2t}\) and \(Y = t^2e^{2t}\) and \(Y = t^3e^{2t}\). By the principle of superposition, the general solution for \(Y\) is a linear combination of these seven.
\[
Y(t) = C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + C_4e^{2t} + C_5te^{2t} + C_6t^2e^{2t} + C_7t^3e^{2t}
\]
Part (c)

The associated homogeneous equation to

$$(D - 2)^3(D + 1)Y = 3e^{2t} - te^{-t} \quad (i)$$

is

$$(D - 2)^3(D + 1)Y_c = 0$$
$$(D^4 - 5D^3 + 6D^2 + 4D - 8)Y_c = 0$$
$$Y_c^{(4)} - 5Y_c''' + 6Y_c'' + 4Y_c' - 8Y_c = 0.$$ 

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form $Y = e^{st}$.

$$Y_c = e^{st} \rightarrow Y_c' = se^{st} \rightarrow Y_c'' = s^2e^{st} \rightarrow Y_c''' = s^3e^{st} \rightarrow Y_c^{(4)} = s^4e^{st}.$$ 

Substitute these expressions into the ODE.

$$s^4e^{st} - 5(s^3e^{st}) + 6(s^2e^{st}) + 4(se^{st}) - 8(e^{st}) = 0$$

Divide both sides by $e^{st}$.

$$s^4 - 5s^3 + 6s^2 + 4s - 8 = 0$$
$$s^4 - 5s^3 + 6s^2 + 4s - 8 = 0$$
$$s = \{-1, 2\}$$

Two solutions to the associated homogeneous equation are $Y = e^{-t}$ and $Y = e^{2t}$. The multiplicity of the $r = 2$ root is 3, so a second and third linearly independent solution can be obtained from the first by including factors of $t$ and $t^2$: $Y = te^{2t}$ and $Y = t^2e^{2t}$. By the principle of superposition, the general solution for $Y$ is a linear combination of these four.

$$Y_c(t) = C_5e^{-t} + C_9e^{2t} + C_{10}te^{2t} + C_{11}t^2e^{2t}$$

Comparing this to the result of part (b), we set $C_1 = 0$, $C_4 = 0$, $C_5 = 0$, and $C_6 = 0$ so that no terms are shared in common.

$$Y(t) = C_2te^{-t} + C_3t^2e^{-t} + C_7t^3e^{2t}$$