Problem 5

In each of Problems 1 through 8, determine the general solution of the given differential equation.

\[ y^{(4)} - 4y'' = t^2 + e^t \]

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y_c^{(4)} - 4y_c'' = 0 \]  

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \rightarrow y'''_c = r^3e^{rt} \rightarrow y_c^{(4)} = r^4e^{rt} \]

Substitute these expressions into the ODE.

\[ r^4e^{rt} - 4(r^2e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^4 - 4r^2 \]

\[ r^2(r^2 - 4) = 0 \]

\[ r = \{-2, 0, 2\} \]

Three solutions to equation (1) are then \( y_c = e^{-2t} \) and \( y_c = e^0 = 1 \) and \( y_c = e^{2t} \). Since the \( r = 0 \) root has a multiplicity of 2, a second linearly independent solution can be obtained by including a factor of \( t \): \( y_c = te^0 = t \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these four.

\[ y_c(t) = C_1e^{-2t} + C_2 + C_3t + C_4e^{2t} \]

On the other hand, the particular solution satisfies

\[ y_p^{(4)} - 4y_p'' = t^2 + e^t \]

The right side has two terms. To account for the first one, we would include \( A + Bt + Ct^2 \) in the trial solution, but because 1 and \( t \) already satisfies \( y_c \), an extra factor of \( t^2 \) is needed. To account for the second one, we will include \( De^t \) in the trial solution. Substitute \( y_p(t) = t^2(A + Bt + Ct^2) + De^t \) in the ODE to determine \( A \) and \( B \) and \( C \) and \( D \).

\[ [t^2(A + Bt + Ct^2) + De^t]^{(4)} - 4[t^2(A + Bt + Ct^2) + De^t]'' = t^2 + e^t \]

Evaluate the derivatives.

\[ (24C + De^t) - 4(2A + 6Bt + 12Ct^2 + De^t) = t^2 + e^t \]

Simplify the left side.

\[ 24C - 8A - 24Bt - 48Ct^2 - 3De^t = t^2 + e^t \]
Match the coefficients to obtain a system of equations for $A$, $B$, $C$, and $D$.

\[
\begin{align*}
24C - 8A &= 0 \\
-24B &= 0 \\
-48C &= 1 \\
-3D &= 1
\end{align*}
\]

Solving this system yields $A = -1/16$, $B = 0$, $C = -1/48$, and $D = -1/3$. As a result, the particular solution is $y_p(t) = t^2(-1/16 - t^2/48) - (1/3)e^t$, and the general solution is

\[
y(t) = C_1e^{-2t} + C_2 + C_3t + C_4e^{2t} - \frac{1}{3}e^t - \frac{t^2}{16} - \frac{t^4}{48}.
\]