**Problem 6**

In each of Problems 1 through 8, determine the general solution of the given differential equation.

\[ y^{(4)} + 2y'' + y = 3 + \cos 2t \]

**Solution**

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y_c^{(4)} + 2y_c'' + y_c = 0 \] \hspace{1cm} (1)

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[
\begin{align*}
y_c &= e^{rt} \\
y_c' &= re^{rt} \\
y_c'' &= r^2e^{rt} \\
y_c''' &= r^3e^{rt} \\
y_c^{(4)} &= r^4e^{rt}
\end{align*}
\]

Substitute these expressions into the ODE.

\[ r^4e^{rt} + 2(r^2e^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^4 + 2r^2 + 1 = 0 \]

\[ (r^2 + 1)^2 = 0 \]

\[ r = \{-i, i\} \]

Two solutions to equation (1) are then \( y_c = e^{-it} \) and \( y_c = e^{it} \). Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of \( t \): \( y_c = te^{-it} \) and \( y_c = te^{it} \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these four.

\[
y_c(t) = C_1e^{-it} + C_2e^{it} + C_3te^{-it} + C_4te^{it} = C_1(\cos t - i \sin t) + C_2(\cos t + i \sin t) + C_3t(\cos t - i \sin t) + C_4t(\cos t + i \sin t) = (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + t(C_3 + C_4)\cos t + t(-iC_3 + iC_4)\sin t = C_5\cos t + C_6\sin t + C_7t\cos t + C_8t\sin t
\]

On the other hand, the particular solution satisfies

\[ y_p^{(4)} + 2y_p'' + y_p = 3 + \cos 2t. \]

The right side has two terms. To account for the first one, we will include \( A \) in the trial solution. To account for the second one, we will include \( B \cos 2t \) in the trial solution. Substitute \( y_p(t) = A + B \cos 2t \) in the ODE to determine \( A \) and \( B \).

\[
(A + B \cos 2t)^{(4)} + 2(A + B \cos 2t)'' + (A + B \cos 2t) = 3 + \cos 2t
\]

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Evaluate the derivatives.

\[(16B \cos 2t) + 2(-4B \cos 2t) + (A + B \cos t) = 3 + \cos 2t\]

Simplify the left side.

\[A + 9B \cos 2t = 3 + \cos 2t\]

Match the coefficients to obtain a system of equations for \(A\) and \(B\).

\[
\begin{align*}
A &= 3 \\
9B &= 1
\end{align*}
\]

Solving this system yields \(A = 3\) and \(B = 1/9\). As a result, the particular solution is \(y_p(t) = 3 + (1/9) \cos 2t\), and the general solution is

\[y(t) = C_5 \cos t + C_6 \sin t + C_7 t \cos t + C_8 t \sin t + 3 + \frac{1}{9} \cos 2t.\]