Problem 9

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

\[ y''' + 4y' = t; \quad y(0) = y'(0) = 0, \quad y''(0) = 1 \]

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'''' + 4y' = 0 \quad (1) \]

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2 e^{rt} \quad \rightarrow \quad y'''_c = r^3 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^3 e^{rt} + 4(re^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^3 + 4r = 0 \]
\[ r(r^2 + 4) = 0 \]
\[ r = \{0, -2i, 2i\} \]

Two solutions to equation (1) are then \( y_c = e^0 = 1 \) and \( y_c = e^{-2it} \) and \( y_c = e^{2it} \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these three.

\[ y_c(t) = C_1 + C_2 e^{-2it} + C_3 e^{2it} \]
\[ = C_1 + (C_2 + C_3) \cos 2t + i(C_2 - C_3) \sin 2t \]
\[ = C_1 + C_4 \cos 2t + C_5 \sin 2t \]

On the other hand, the particular solution satisfies

\[ y'''_p + 4y'_p = t. \]

To account for the inhomogeneous term, we would include \( A + Bt \) in the trial solution, but because a constant satisfies equation (1), an extra factor of \( t \) is needed. Substitute \( y_p(t) = t(A + Bt) \) in the ODE to determine \( A \) and \( B \).

\[ [t(A + Bt)]''' + 4[t(A + Bt)]' = t \]

Evaluate the derivatives.

\[ (0) + 4(A + 2Bt) = t \]
Simplify the left side.

\[ 4A + 8Bt = t \]

Match the coefficients to obtain a system of equations for \( A \) and \( B \).

\[
\begin{align*}
4A &= 0 \\
8B &= 1
\end{align*}
\]

Solving this system yields \( A = 0 \) and \( B = 1/8 \). As a result, the particular solution is \( y_p(t) = t(t/8) \), and the general solution is

\[ y(t) = C_1 + C_4 \cos 2t + C_5 \sin 2t + \frac{t^2}{8}. \]

Differentiate it with respect to \( t \) twice.

\[
\begin{align*}
y'(t) &= -2C_4 \sin 2t + 2C_5 \cos 2t + \frac{t}{4} \\
y''(t) &= -4C_4 \cos 2t - 4C_5 \sin 2t + \frac{1}{4}
\end{align*}
\]

Now apply the initial conditions to determine \( C_1 \), \( C_4 \), and \( C_5 \).

\[
\begin{align*}
y(0) &= C_1 + C_4 = 0 \\
y'(0) &= 2C_5 = 0 \\
y''(0) &= -4C_4 + \frac{1}{4} = 1
\end{align*}
\]

Solving this system of equations yields \( C_1 = 3/16 \), \( C_4 = -3/16 \), and \( C_5 = 0 \). Therefore,

\[ y(t) = \frac{3}{16} - \frac{3}{16} \cos 2t + \frac{t^2}{8}. \]