# Problem 2

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' - y' = t$$

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**Solution**

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - re^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for $y_c$ is a linear combination of these three.

$$y_c(t) = C_1 e^{-t} + C_2 + C_3 e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{-t} + C_2(t) + C_3(t)e^t$$

Substitute this formula into equation (2).

$$[C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]''' - [C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]' = t$$

Evaluate the derivatives.

$$[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t]'' - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t] = t$$

If we set $C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0$, then this equation simplifies to

$$[-C_1(t)e^{-t} + C_3(t)e^t]'' - [-C_1(t)e^{-t} + C_3(t)e^t] = t$$

$$[-C_1'(t)e^{-t} + C_1(t)e^{-t} + C_3'(t)e^t + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = t.$$

If we set $-C_1'(t)e^{-t} + C_3'(t)e^t = 0$, then this equation simplifies to

$$[C_1(t)e^{-t} + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = t$$

$$[C_1'(t)e^{-t} - \cancel{C_1(t)e^{-t}} + C_3'(t)e^t + \cancel{C_3(t)e^t}] - [-\cancel{C_1(t)e^{-t}} + \cancel{C_3(t)e^t}] = t$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = t.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \tag{3}$$

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \tag{4}$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = t \tag{5}$$

Start by solving equation (4) for $C_1'(t)$.

$$C_1'(t) = e^{2t}C_3'(t)$$

and then plugging it in to equation (5).

$$[e^{2t}C_3'(t)]e^{-t} + C_3'(t)e^t = t$$

$$2C_3'(t)e^t = t$$

$$C_3'(t) = \frac{1}{2}te^{-t}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_3(t) = -\frac{1}{2}(t+1)e^{-t}$$

Substitute this back into equation (4) to get $C_1(t)$.

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \left[\frac{1}{2}te^{-t}\right]e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \frac{1}{2}t = 0$$

$$C_1'(t) = \frac{1}{2}te^t$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_1(t) = \frac{1}{2}(t-1)e^t$$

Substitute this result along with $C_3(t)$ into equation (3) to obtain $C_2(t)$.

$$C_1'(t)e^{-t}+C_2'(t)+C_3'(t)e^t = 0 \quad \rightarrow \quad \left[\frac{1}{2}te^t\right]e^{-t}+C_2'(t)+\left[\frac{1}{2}te^{-t}\right]e^t = 0 \quad \rightarrow \quad \frac{1}{2}t+C_2'(t)+\frac{1}{2}t = 0$$

$$C_2'(t) = -t$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_2(t) = -\frac{t^2}{2}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)e^{-t} + C_2(t) + C_3(t)e^t \\ &= \left[\frac{1}{2}(t-1)e^t\right]e^{-t} + \left(-\frac{t^2}{2}\right) + \left[-\frac{1}{2}(t+1)e^{-t}\right]e^t \\ &= \frac{1}{2}(t-1) - \frac{t^2}{2} - \frac{1}{2}(t+1) \\ &= -\frac{t^2}{2} - 1. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{-t} + C_2 + C_3e^t - \frac{t^2}{2} - 1 \\ &= C_1e^{-t} + C_4 + C_3e^t - \frac{t^2}{2}, \end{aligned}$$

where a new constant $C_4$ is used for $C_2 - 1$.