Problem 5

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' - y'' + y' - y = e^{-t}\sin t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c'' + y_c' - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - r^2e^{rt} + re^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$\begin{gathered} r^3 - r^2 + r - 1 = 0 \\ (r-1)(r^2+1) = 0 \\ r = \{1, -i, i\} \end{gathered}$$

Three solutions to equation (1) are then $y_c = e^t$ and $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1e^t + C_2e^{-it} + C_3e^{it} \\ &= C_1e^t + C_2(\cos t - i\sin t) + C_3(\cos t + i\sin t) \\ &= C_1e^t + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\ &= C_1e^t + C_4\cos t + C_5\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p'' + y_p' - y_p = e^{-t}\sin t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t$$

Substitute this formula into equation (2).

$$\begin{aligned} &[C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t]''' - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t]'' \\ &\qquad + [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t]' - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = e^{-t}\sin t \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}
&[C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]'' \\
&\quad - [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]' \\
&\quad + [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t] \\
&\quad - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = e^{-t}\sin t
\end{aligned}$$

If we set $C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0$, then this equation simplifies to

$$\begin{aligned}
&[C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]'' - [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]' \\
&\quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = e^{-t}\sin t
\end{aligned}$$

$$\begin{aligned}
&[C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t]' \\
&\quad - [C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t] \\
&\quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = e^{-t}\sin t.
\end{aligned}$$

If we set $C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0$, then this equation simplifies to

$$\begin{aligned}
&[C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t]' - [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t] \\
&\quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = e^{-t}\sin t
\end{aligned}$$

$$\begin{aligned}
&[C_1'(t)e^t + \cancel{C_1(t)e^t} - C_4'(t)\cos t + \cancel{C_4(t)\sin t} - C_5'(t)\sin t - C_5(t)\cos t] \\
&\quad - [\cancel{C_1(t)e^t} - \cancel{C_4(t)\cos t} - C_5(t)\sin t] \\
&\quad + [\cancel{C_1(t)e^t} - \cancel{C_4(t)\sin t} + C_5(t)\cos t] - [\cancel{C_1(t)e^t} + \cancel{C_4(t)\cos t} + C_5(t)\sin t] = e^{-t}\sin t \\
&C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = e^{-t}\sin t.
\end{aligned}$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0 \tag{3}$$
$$C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0 \tag{4}$$
$$C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = e^{-t}\sin t \tag{5}$$

Add the respective sides of equations (3) and (5) together.

$$2C_1'(t)e^t = e^{-t}\sin t$$

$$C_1'(t) = \frac{1}{2}e^{-2t}\sin t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1(t) = -\frac{1}{10}e^{-2t}(2\sin t + \cos t)$$

Subtract the respective sides of equation (4) from those of equation (3). Also, subtract the respective sides of equation (5) from those of equation (3).

$$\begin{aligned}
C_4'(t)(\cos t + \sin t) + C_5'(\sin t - \cos t) &= 0 \\
2C_4'(t)\cos t + 2C_5'(t)\sin t &= -e^{-t}\sin t
\end{aligned}$$

Solve this first equation for $C_4'(t)$

$$C_4'(t) = \frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \tag{6}$$

and then plug it into the second equation.

$$2\left[\frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t)\right]\cos t + 2C_5'(t)\sin t = -e^{-t}\sin t$$

Divide both sides by 2 and factor $C_5'(t)$.

$$C_5'(t)\left(\frac{\cos t - \sin t}{\cos t + \sin t}\cos t + \sin t\right) = -\frac{1}{2}e^{-t}\sin t$$

$$C_5'(t)\left[\frac{\cos t - \sin t}{\cos t + \sin t}\cos t + \frac{\sin t(\cos t + \sin t)}{\cos t + \sin t}\right] = -\frac{1}{2}e^{-t}\sin t$$

$$C_5'(t)\left(\frac{\cos^2 t - \cancel{\sin t\cos t} + \cancel{\sin t\cos t} + \sin^2 t}{\cos t + \sin t}\right) = -\frac{1}{2}e^{-t}\sin t$$

$$C_5'(t)\left(\frac{1}{\cos t + \sin t}\right) = -\frac{1}{2}e^{-t}\sin t$$

$$C_5'(t) = -\frac{1}{2}e^{-t}\sin t(\cos t + \sin t)$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_5(t) = \frac{1}{20}e^{-t}(\cos 2t + 3\sin 2t + 5)$$

Now substitute the previous equation into equation (6).

$$\begin{aligned} C_4'(t) &= \frac{\cos t - \sin t}{\cos t + \sin t}\left[-\frac{1}{2}e^{-t}\sin t(\cos t + \sin t)\right] \\ &= -\frac{1}{2}e^{-t}\sin t(\cos t - \sin t) \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_4(t) = \frac{1}{20}e^{-t}(3\cos 2t - \sin 2t - 5)$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t \\ &= \left[-\frac{1}{10}e^{-2t}(2\sin t + \cos t)\right]e^t + \left[\frac{1}{20}e^{-t}(3\cos 2t - \sin 2t - 5)\right]\cos t + \left[\frac{1}{20}e^{-t}(\cos 2t + 3\sin 2t + 5)\right]\sin t \\ &= -\frac{1}{10}e^{-t}(2\sin t + \cos t) + \frac{1}{20}e^{-t}(3\cos t\cos 2t - \cos t\sin 2t - 5\cos t) + \frac{1}{20}e^{-t}(\sin t\cos 2t + 3\sin t\sin 2t + 5\sin \\ &= \frac{1}{20}e^{-t}(-4\sin t - 2\cos t + 3\cos t\cos 2t - \cos t\sin 2t - 5\cos t + \sin t\cos 2t + 3\sin t\sin 2t + 5\sin t) \\ &= \frac{1}{20}e^{-t}[\sin t - 7\cos t + 3\cos t(1 - 2\sin^2 t) - \cos t(2\sin t\cos t) + \sin t(2\cos^2 t - 1) + 3\sin t(2\sin t\cos t)] \\ &= \frac{1}{20}e^{-t}(\cancel{\sin t} - 7\cos t + 3\cos t - \cancel{6\sin^2 t\cos t} - \cancel{2\cos^2 t\sin t} + \cancel{2\cos^2 t\sin t} - \cancel{\sin t} + \cancel{6\sin^2 t\cos t}) \\ &= \frac{1}{20}e^{-t}(-4\cos t) \\ &= -\frac{1}{5}e^{-t}\cos t. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 e^t + C_4 \cos t + C_5 \sin t - \frac{1}{5} e^{-t} \cos t. \end{aligned}$$